3. Music

i) For the tubular bells we can find the speed of sound in steel using dimensional analysis. It can depend on the young modulus E and on the density ρ. We find that $v \propto \sqrt{\frac{E}{\rho}} \propto \sqrt{\frac{EL^3}{m}}$, where L is the characteristic length of the bell. The speed, frequency and wavelength are related by $f = \frac{v}{\lambda}$. For a standing wave to exist on the bell, the wavelength must be proportional to it's characteristic length, because it has anti-nodes at the ends: $\lambda \propto L$. We get $f \propto \frac{\sqrt{L^3}}{L} = \sqrt{L}$. Due to thermal expansion, the lengths scale by $(1+\alpha\Delta T)$, where α is the coefficient of thermal expansion. Using this we have $f = f_0\sqrt{1+\alpha\Delta T} \approx 330.04\,\text{Hz}$. This is a very small change from $f_0 = 330\,\text{Hz}$.

For the pipe organ we also have $f = \frac{v}{\lambda}$. The sound is generated by a standing wave in the air column. The speed v in the formula is the speed of sound in the air. As in the previous case, the wavelength has to be proportional to the length of the pipe for a standing wave to exist and the length scales by $(1+\alpha T)$. Since $\alpha T \approx 2.4 \times 10^{-4}$ is much smaller than the change in the speed of sound, we can ignore it. Using the given formula for the speed of sound, we have $f = f_0\sqrt{1+\frac{45}{273.15}}/\sqrt{1+\frac{25}{273.15}} \approx 341\,\text{Hz}$. This is a significant change.

For the guitar string the speed of the wave can depend on the mass m of the string, the length L and the tension T. Using dimensional analysis we get $v \propto \sqrt{\frac{LT}{m}}$. A more accurate derivation will show that the constant of proportionality is 1, which would be the simplest guess. The lowest frequency on the string has a wavelength of $2L$, because the wave has to have nodes at the ends of the string. The tension is given by $T = ES\Delta L/L$, where ΔL is the change in length of the string due to the tension and S is the area of the cross section of the string. Putting it all together:

$$f = \frac{\sqrt{\frac{LT}{m}}}{2L} = \frac{\sqrt{\frac{LES\,\Delta L/L}{\rho SL}}}{2L} = \frac{\sqrt{\frac{E\,\Delta L/L}{\rho}}}{2L}.$$

Let's find $\frac{\Delta L}{L} = \frac{\rho}{E}(2Lf)^2 \approx 7.3\times10^{-3}$. When the temperature increases and the string stretches, the relative lengthening $\frac{\Delta L}{L}$ will decrease by αT,

$$\frac{\Delta L}{L} \to \frac{\Delta L}{L} - \alpha T = \frac{\Delta L}{L}\left(1 - \frac{\alpha T}{\Delta L/L}\right) = \frac{\Delta L}{L}\left(1 - 3.3\times10^{-2}\right).$$

This proportional change is larger than the changes in the other factors (the lengths get scaled by $(1+\alpha\Delta T)$), which we can ignore. We get $f = f_0\sqrt{1-3.3\times10^{-2}} \approx 325\,\text{Hz}$. This is also a significant change when the instruments are playing together.

ii) For the tubular bells and for the guitar the frequency is determined by the properties and the speed of sound of the metal and therefore doesn't change in the heliox. For the pipe organ, the sound is generated directly by the vibrating air column in the pipe and changing the air to heliox will change the frequency. The wavelength is determined by the length of the pipe and since the frequency is $f = \frac{v}{\lambda}$, it will change proportionally with the speed of sound, $f = 1.7 \cdot f_0 \approx 561\,\text{Hz}$. This is a very significant change.