4. Dimmer

i) Denote $\omega = 2\pi f$. Adding an imaginary component to the voltages $u = Ue^{j\omega t}$ and $u_C = U_C e^{j(\omega t+\varphi)}$ (with j being the imaginary unit), there is a relation $u_C = \frac{uz_C}{R+z_C}$, where the capacitor's impedance $z_C = 1/(j\omega C)$. Calculating, we get

$$k \equiv \frac{z_C}{R+z_C} = \frac{1}{j\omega RC+1} = \frac{1}{\left[\sqrt{1+(\omega RC)^2} \times e^{j \arctan(\omega RC)}\right]}.$$

Therefore $U_C = U|k| = U/\sqrt{1+(2\pi fRC)^2}$ and $\varphi = \arg k = -\arctan(2\pi fRC)$. (The answers could also be obtained by analyzing a vector diagram.)

ii) The diac starts to conduct when the voltage on the capacitor rises to U_b. This voltage is then applied to the series connection of R_t and the diac, whose voltage has dropped to U_d. Therefore the current through the diac is $(U_b - U_d)/R_t$ and the inequality is $R_t I_t < U_b - U_d$.

iii) At t_0, the voltage on the capacitor becomes larger than U_b. Thus, $U_C \sin(2\pi f t_0 + \varphi) = U_b$ and $t_0 = [\arcsin(U_b/U_C) - \varphi]/(2\pi f)$.

iv) The lamp is lit at phase $2\pi f t_0$ and turned off at phase π. The power is proportional to the voltage squared, giving the ratio

$$r = \frac{\int_0^\pi \sin^2\alpha \,\mathrm{d}\alpha}{\int_{2\pi f t_0}^\pi \sin^2\alpha \,\mathrm{d}\alpha}.$$

Because $\int \sin^2\alpha \,\mathrm{d}\alpha = \int \frac{1-\cos(2\alpha)}{2}\,\mathrm{d}\alpha = \frac{\alpha}{2} - \frac{\sin(2\alpha)}{4}$, we get

$$\begin{aligned} r &= \frac{\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(\frac{0}{2} - \frac{\sin(0)}{4}\right)}{\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(\frac{2\pi f t_0}{2} - \frac{\sin(4\pi f t_0)}{4}\right)} \\ &= \frac{1}{1 - 2ft_0 + \frac{\sin(4\pi f t_0)}{2\pi}}. \end{aligned}$$