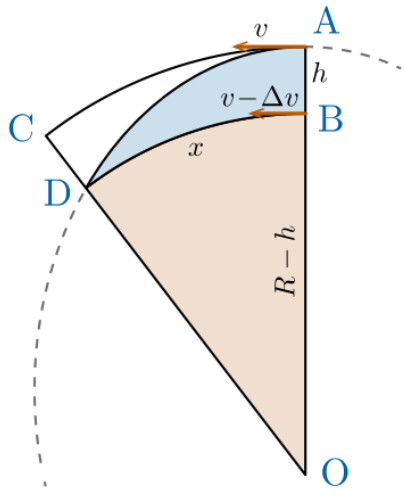
9. Deflection on Falling

i) The Earth is rotating with angular velocity $\omega = \frac{2\pi}{T}$, where $T = 24\,\mathrm{h}$. The velocities at the top and bottom of the shaft are $v_t = \omega R$ and $v_b = \omega(R-h)$, where R is radius of Earth. The difference of the velocities is thus $\Delta v = v_t - v_b = \omega h \approx 7.3\,\mathrm{mm/s}$.

ii) The time of free fall can be found from the relation $h = gt^2/2$, giving $t = \sqrt{2h/g}$. Thus, the horizontal displacement is simply $\Delta x = \Delta v t = \omega h t = \omega\sqrt{2h^3/g} \approx 33\,\mathrm{mm}$.

iii) There are at least three different approaches to this problem; one is using the angular momentum conservation law, second one is based on Kepler's laws (given later below), and the third one — which we don't consider here — is based on Coriolis force formula.

Consider the rotation speed ω' of the radius vector drwan from the falling body to the centre of Earth, and let us compare this speed with the rotation speed of Earth ω. The angular momentum of the falling body is conserved, hence $\omega' r^2 = \omega R^2$. We can substitute $r = R - y$, where y is the current depth, and approximate $\omega' = \omega\left(\frac{R}{r}\right)^2 \approx \omega\left(1 + 2\frac{y}{R}\right)$. Therefore, the horizontal displacement speed in the Earth's frame of reference $v_h = (\omega' - \omega)r \approx 2\omega y R$. Finally, horizontal dipslacement is found as $\Delta x = \int v_h\,\mathrm{d}t = \int (2\omega y R/v_v)\,\mathrm{d}y$, where the vertical falling speed $v_v = \sqrt{2gy}$. So, we find $\Delta x = \frac{2}{3}R\omega\sqrt{2h^3/g} \approx 22$ m.



Now, let obtain the same result using the Kepler's laws. Consider the trajectory of the steel ball as seen in a non-rotating frame of reference. Although it is a thin ellipse, we have drawn the figure out of scale in order the facilitate the calculation of areas. The steel ball is released from point A and it hits the bottom of the shaft at point D, at distance x from B, the location of bottom at the start of fall. As the falling time is still $t = \sqrt{2h/g}$, the location of the bottom travels $x' = (v - \Delta v)t = vt - \omega h t$ during the fall. Thus, the horizontal displacement of the landing point is simply $\Delta x = x - x'$.

Now, the distance x can be found using the Kepler's second law, stating that the area covered by radius vector per unit time $\Delta S/\Delta t$ is constant, which is a manifestation of conservation of angular momentum $\Delta S/\Delta t = L/2m = rv_\perp/2$. (The latter relation could easily be obtained by observing a circular orbit.) For our steel ball, $L/2m = Rv/2 = \omega R^2/2$. The area covered by the steel ball can be calculated as the sum of the segment OBD and the region ADB. Keeping in mind that $x \ll R$, the segment OBD is simply a triangle with area $x(R-h)/2$. Likewise, the region ACDB is approximately a rectangle of area xh and knowing that a parabola divides the area of its surrounding rectangle into proportions 1/3 and 2/3, we conclude that the area of region ADB is $2xh/3$. Thus, from the Kepler's second law:

$$S = \frac{1}{2}x(R-h) + \frac{2}{3}xh = \frac{1}{2}vRt,$$
$$x = \frac{vt}{1 + \frac{1}{3}\frac{h}{R}} \approx vt - \frac{1}{3}\omega h t.$$

Finally, the horizontal displacement $\Delta x = x - x' = \frac{2}{3}\omega h t \approx 22\,\mathrm{mm}$. (Note that the naive answer overestimated the correct one by 50%.)