4. GRAVITATIONAL WAVES

i) Black holes will merge at this distance.

$$r_{\min} = r_{s1} + r_{s2} = \frac{2G(m_1 + m_2)}{c^2}$$

Substituting for r in the power equation, we get the maximum power for a system with masses m_1 and m_2.

$$P_{\max}(m_1, m_2) = \frac{1}{5}\frac{c^5}{G}\frac{(m_1 m_2)^2}{(m_1 + m_2)^4}$$

Now let us consider the effect of masses. Let us define $a := \frac{(m_1 m_2)^2}{(m_1+m_2)^4}$. The intuition tells us, that more power should be radiated off a bigger system, thus the masses should tend to infinity for maximum power.

However the limit $\lim_{m_1=m_2\to\infty} a = 1$ suggests otherwise. So does the dimensional analysis: the dimension of $[a] = \frac{(M\cdot M)^2}{M^4} = 1$ means that the absolute values of the masses don't contribute to the equation. This is intuitively explained by the proportionally increasing Schwarzschild radii.

However the mass distribution does matter, since a isn't the same for all m_1, m_2 either. Let us determine the maximum value it can take. Let us redefine it in terms of the total mass $M := m_1 + m_2$ and the mass distribution $k := \frac{m_1}{M}$. Then

$$\begin{aligned}
m_1 &= kM \\
m_2 &= (1-k)M \\
\Rightarrow a &= \frac{(m_1 m_2)^2}{(m_1 + m_2)^4} = \frac{(kM(k-1)M)^2}{M^4} = [k(k-1)]^2 \\
\frac{d}{dk}[k(k-1)]^2 &= 0 \Leftrightarrow k = \tfrac{1}{2} \Leftrightarrow m_1 = m_2 \\
\Rightarrow a_{\max} &= \left[\tfrac{1}{2}\left(1 - \tfrac{1}{2}\right)\right]^2 = \tfrac{1}{16}
\end{aligned}$$

So that finally

$$P_{\max} = \frac{1}{5}\frac{c^5}{G}\frac{1}{16} = \frac{1}{80}\frac{c^5}{G} = 4.5 \times 10^{50}\,\mathrm{W}$$

Interestingly enough, the peak power emitted in gravitational radiation by any merging binary system will be the same. $\frac{c^5}{G}$ is also called luminosity of the universe.

ii) $U = \frac{1}{2}k\Delta\ell^2$, energy stored in a uniformly deformed body

$$\begin{aligned}
k &= \frac{F}{\Delta\ell} \\
F &= \sigma S \\
\Delta\ell &= \varepsilon\ell \\
\sigma &= \varepsilon E \\
u &= \frac{U}{V}, \text{ where } V = S\ell \text{ is the volume} \\
\Rightarrow u &= \frac{U}{V_0} = \tfrac{1}{2}E\varepsilon^2
\end{aligned}$$

iii) $[G] = L^3M^{-1}T^{-2}$, from e.g. $F = \frac{GMm}{r^2}$

$$\begin{aligned}
[c] &= LT^{-1} \\
[f] &= T^{-1} \\
[E] &= ML^{-1}T^{-2}, \text{ from e.g. } \sigma = \varepsilon E
\end{aligned}$$

let $E = G^\alpha c^\beta f^\gamma$. Solving for each dimension yields

$$\begin{aligned} M\colon 1 &= -\alpha \quad \Rightarrow \alpha = -1 \\ L\colon -1 &= 3\alpha + \beta \quad \Rightarrow \beta = 2 \\ T\colon -2 &= -2\alpha - \beta - \gamma \quad \Rightarrow \gamma = 2 \\ \Rightarrow E &= \frac{c^2}{G} f^2 \end{aligned}$$

iv) The gravitational wave spreads as a spherical wave centered at the source. It's wavefront occupies a thin spherical shell with the surface area $S = 4\pi z^2$ and its associated infinitesimal volume is $dV = S\,dz$, where $dz = c\,dt$ for a wave traveling with the speed of light c. The power P and energy density u at a distance z is coupled through $dE = P\,dt = u\,dV \Leftrightarrow z = \sqrt{\frac{P}{4\pi u c}}$.

Using the models from

$$\begin{aligned} P &= \frac{1}{80}\frac{c^5}{G} \\ u &= \frac{U}{V_0} = \frac{1}{2} E \varepsilon^2 \\ E &= \frac{c^2}{G} f^2 \end{aligned}$$

we finally get

$$z = \frac{1}{\sqrt{160\pi}} \frac{c}{\varepsilon f}$$