1. Gravitational racing

i) (a) Since all three bodies move along the same trajectory, they must be $\frac{T}{3}$ away from each other at any moment of time. Thus, it takes $\frac{T}{3}$ to get from O_2 to O.

(b) From symmetry, time taken to go from P to O must be $\frac{T}{4}$. Furthermore, it takes $\frac{T}{3}$ to get from P_2 to P and from O to O_3. This means that it takes $\frac{T}{3}+\frac{T}{3}+\frac{T}{4}=\frac{11T}{12}$ to get from P_2 to O_3 or $T-\frac{11T}{12}=\frac{T}{12}$ to get from O_3 to P_2.

ii) Since there are no external forces at play, the centre of mass of the three body system must stay in place and, due to symmetry, be located at O. Thus, $\vec{r}_1+\vec{r}_2+\vec{r}_3=0$, where $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ are position vectors from O. Differentiating,

$$\vec{v}_1+\vec{v}_2+\vec{v}_3=0. \tag{1}$$

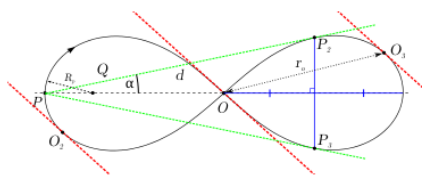
iii) The total angular momentum is conserved. Thus, we can find the angular momentum at a moment of time that's most convenient for us, such as the configuration when one of the bodies is at O. Due to symmetry, $\vec{r}_2=-\vec{r}_3$ and $\vec{v}_2=\vec{v}_3$. The total angular momentum is then

$$\begin{aligned} J &= m\vec{r}_1\times\vec{v}_1+m\vec{r}_2\times\vec{v}_2+m\vec{r}_3\times\vec{v}_3= \\ &= m(\vec{r}_2\times\vec{v}_2+\vec{r}_3\times\vec{v}_3)=m(\vec{r}_2\times\vec{v}_2-\vec{r}_2\times\vec{v}_2)=0. \end{aligned}$$

iv) Using equation ?? and the fact that $\vec{v}_2=\vec{v}_3$, $\vec{v}_1+2\vec{v}_2=0$ must hold. Thus, $\vec{v}_2=\frac{1}{2}\vec{v}_1$. Note that this can only happen when $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ are parallel. Since the velocities of the bodies are tangent to the trajectory at their locations, the trajectories of the bodies must have the same slope at O, O_2 and O_3. This gives a convenient way of reconstructing the positions of O_2 and O_3 using a ruler as shown in the figure in red.

v) We can use symmetry to deduce that $r_{2,x}=r_{3,x}$ and $r_{2,y}=-r_{3,y}$. Additionally, $r_{1,x}+r_{2,x}+r_{3,x}=0$, so $r_{2,x}=r_{3,x}=-\frac{1}{2}r_{1,x}$. This gives a simple way to find the x-coordinates of P_2 and P_3 and reconstruct their positions shown in blue in the figure.

Let's apply the conservation of angular momentum at point P. Due to symmetry, the velocity vectors at P_2 and P_3 must intersect at the x-axis. For the total angular momentum to be 0, they must intersect at P. Therefore, P_2 and P_3 have tangents that pass through P. P_2 and P_3 can then be reconstructed as shown in green in the figure.



vi) The general idea is to apply energy conservation in both configurations and use the radius of curvature at P to find the value of Gm. From the figure, $R_p=41.7$, $r_0=145.2$, $d=242$ and $\alpha=11.6\,\text{deg}$ (the units for distance are arbitrary, only the ratios carry any meaning).

The total energy at O is

$$E=\frac{mv_{1,o}^2}{2}+\frac{mv_{2,o}^2}{2}+\frac{mv_{3,o}^2}{2}-\frac{Gm^2}{r_{12,o}}-\frac{Gm^2}{r_{23,o}}-\frac{Gm^2}{r_{31,o}}.$$

We have $v_{1,o}=2v_{2,o}=2v_{3,o}=v_o$ and $r_{12,o}=r_{31,o}=\frac{1}{2}r_{23,o}=r_o$. Thus,

$$E=\frac{3mv_o^2}{4}-\frac{5Gm^2}{2r_o}. \tag{2}$$

Total Energy at P is

$$E=\frac{mv_{1,p}^2}{2}+\frac{mv_{2,p}^2}{2}+\frac{mv_{3,p}^2}{2}-\frac{Gm^2}{r_{12,p}}-\frac{Gm^2}{r_{23,p}}-\frac{Gm^2}{r_{31,p}}.$$

Additionally, $r_{12,p} = r_{31,p} = d$ and $r_{23,p} = 2d\sin\alpha$. Furthermore, applying equation ?? on the y-axis, $v_{1,p} - v_{2,p}\sin\alpha - v_{3,p}\sin\alpha = 0$. Thus, $v_{1,p} = v_p = 2v_{2,p}\sin\alpha = 2v_{3,p}\sin\alpha$ since $v_{2,p} = v_{3,p}$. The total energy at P is then

$$E = \frac{mv_p^2}{2}\left(1 + \frac{1}{2\sin^2\alpha}\right) - \frac{Gm^2}{d}\left(2 + \frac{1}{2\sin\alpha}\right) = \\ = 6.68\, mv_p^2 - 4.49\,\frac{Gm^2}{d}. \tag{3}$$

When a body is at P, the gravitational force is equal to the centrifugal force. This means that

$$\frac{mv_p^2}{R_p} = 2\frac{Gm^2}{d^2}\cos\alpha = 1.96\,\frac{Gm^2}{d^2},$$

$$Gm = 0.510\,\frac{v_p^2 d^2}{R_p}. \tag{4}$$

Combining equations ??, ?? and ?? gives

$$\frac{3v_o^2}{4} = 6.68\, v_p^2 + 0.510\,\frac{v_p^2 d^2}{R_p}\left(\frac{5}{2r_o} - \frac{4.49}{d}\right),$$

rearranging,

$$\frac{v_o}{v_p} = \sqrt{\frac{4}{3}\left(6.68 + 0.510\,\frac{d^2}{R_p}\left(\frac{5}{2r_o} - \frac{4.49}{d}\right)\right)} = 2.8.$$