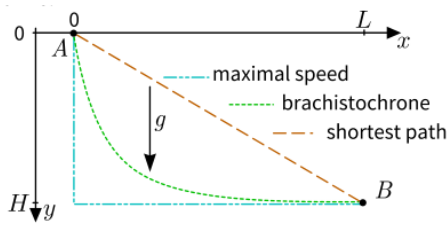
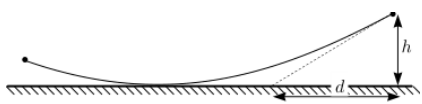
5. Brachistochrone (10 points) — *Rūdolf Treilis.* Consider points A and B separated by height H in the vertical direction and distance L in the horizontal direction, placed in a gravitational field g as shown in the figure below. A point mass can slide along a rail of fixed shape frictionlessly (including taking 90°-turns) from A to B. The brachistochrone curve is the curve minimizing the total travel time.



i) *(2 points)* Calculate the total travel time for the "maximal speed" and "shortest path" trajectories. Find the ratio $\frac{L}{H}$ for which the two are equal.

ii) *(2 points)* According to Fermat's principle, light ray travels from one point to another along the path of shortest travel time. Suppose that in a certain medium, a light ray can propagate from A to B along the brachistochrone curve shown in the figure above. Find the refractive index $n = n(x, y)$ as a function of the coordinates x and y for this medium if $n(L, H) = 1$.

iii) *(2 points)* Show that the path of a light ray traveling in a medium with a variable refractive index $n(x, y) \equiv n(y)$ satisfies the differential equation $\frac{dy}{dx} = \sqrt{C \cdot n(y)^2 - 1}$, where C is a constant determined by boundary conditions.



iv) *(2 points)* The obtained equation can explain mirages, which occur when the index of refraction increases with height. Consider a light ray coming from the sky that grazes the surface of the Earth ($y = 0$) and hits the eye of an observer at height h (for this task choose the y-axis in the opposite direction – bottom of the page to top). If the refractive index varies as $n(y) = n_0(1 + \alpha y)$ with n_0 and α constant, find an expression for the apparent distance that the ray of light is emanating from d.

v) *(2 points)* Solving the equations derived in parts ii) and iii), one may show that the brachistochrone curve is actually a segment of a cycloid. A cycloid is the curve traced by a fixed point on the rim of a circular wheel as it rolls along a straight line without slipping. For the special case $\frac{L}{H} = \frac{\pi}{2}$ find the minimum travel time $t_{\min}$ between A and B.