6. Self-gravitating gas (10 points) — *Eero Vaher (v: Jaan Kalda).* Consider a ball of monoatomic ideal gas at temperature T which keeps a spherically symmetric mechanically stable stationary shape due to its own gravity field. Let the total mass of the gas be M_0, and its molar mass μ. We shall describe the radial mass distribution in terms of the total mass $M = M(r)$ inside a sphere of radius r, and the pressure $p = p(r)$ as a function of distance r from the centre of the spherical gas cloud.

i) *(2 points)* State the mechanical equilibrium condition for a small parcel of gas of mass m and volume v at distance r from the centre of the cloud in terms of the local pressure gradient $p'(r) = \frac{\mathrm{d}p}{\mathrm{d}r}$, the local value of $M(r)$, and the appropriate constants of nature. Simplify your expression by noticing that $m/v = \rho$ is the local density of the gas.

ii) *(2 points)* Show that the total thermal energy of the gas can be expressed as

$$U = -\alpha \int V \,\mathrm{d}p,$$

where $V = \frac{4}{3}\pi r^3$, and α is a numerical factor, and the integral is taken from the centre of the cloud, to very far distances from the centre, where the pressure is negligibly small. Find the value of α.

iii) *(3 points)* Based on your previous results, show that $U = -\beta E_G$, where E_G is the gravitational potential enery of the gas, and $\beta < 1$ is a positive numerical factor. Find the value of β.

iv) *(1 point)* How will the temperature and characteristic radius of the cloud change in time due to heat radiation? Give a qualitative motivated answer. (The characteristic radius R_c can be defined as such a radius that half of the mass of the cloud is inside the sphere of radius R_c.)

v) *(2 points)* Consider now a similar fully ionized plasma cloud. Assume that plasma is a macroscopically neutral mixture of electrons and protons. What is the proportionality factor between the total gravitational energy and total thermal energy for the plasma cloud?