1. SATELLITE (8 points) — *Taavet Kalda.*

i) *(1 point)* From the energy conservation law,

$$\frac{mv_m^2}{2} - \frac{GM_\oplus m}{r_\oplus} = 0,$$

hence

$$v_m = \sqrt{\frac{2GM_\oplus}{r_\oplus}}.$$

ii) *(2 points)* Let the speed of the satellite just after leaving Earth's gravitational field be v_1. From energy conservation,

$$\frac{mv_0^2}{2} - \frac{GM_\oplus m}{r_\oplus} = \frac{mv_1^2}{2},$$

where $M_\oplus$ is Earth's mass. Furthermore, we have $g = \frac{GM_\oplus}{r_\oplus^2}$. Thus,

$$v_1 = \sqrt{v_0^2 - \frac{2GM_\oplus}{r_\oplus}} = \sqrt{v_0^2 - 2gr_\oplus}.$$

iii) *(2.5 points)* The average solar irradiance can be expressed as

$$I_{\text{avg}} = \frac{1}{T}\int_0^T I(t)\,\mathrm{d}t,$$

where $I(t)$ the solar irradiance at time t. We can express the solar irradiance as $I(t) = \frac{L_\odot}{4\pi r(t)^2}$. Furthermore, it might be more convenient to integrate over the angle instead of time so we can use $\mathrm{d}t = \frac{\mathrm{d}\alpha}{\omega(\alpha)}$, where ω is the angular velocity of the satellite. This yields

$$I_{\text{avg}} = \frac{1}{T}\int_0^{2\pi} \frac{L_\odot}{4\pi r^2}\frac{\mathrm{d}\alpha}{\omega} = \frac{L_\odot}{4\pi T}\int_0^{2\pi}\frac{\mathrm{d}\alpha}{\omega r^2}.$$

Note that the denominator in the integrand is very similar to the angular momentum of the satellite. Indeed, the angular momentum is $J = mvr = m\omega r^2 = \text{Const}$. Thus,

$$I_{\text{avg}} = \frac{L_\odot}{4\pi T}\int_0^{2\pi}\frac{\mathrm{d}\alpha\, m}{J} = \frac{L_\odot m}{4\pi TJ}\int_0^{2\pi}\mathrm{d}\alpha = \frac{L_\odot m}{2TJ}.$$

iv) *(2.5 points)* Since $L_\odot$ and m are constant, we need to minimise the quantity TJ. Note that the minimal angular momentum corresponds to the case when the satellite is launched directly opposite to the motion of Earth. It turns out that this also corresponds to the minimal orbital period. Let $\vec{v}_2$ be the satellite's velocity in Sun's frame. Then $\vec{v}_2 = \vec{v}_1 - \vec{v}_\oplus$, where $v_\oplus = \sqrt{\frac{GM_\odot}{R_\oplus}}$ is Earth's velocity. For convenience, let's write $x = \frac{v_2}{v_\oplus}$. Consider the total energy of an elliptical orbit $E_{\text{tot}} = -\frac{GM_\odot m}{2a}$. On the other hand, the total energy is $E_{\text{tot}} = \frac{mv_2^2}{2} - \frac{GM_\odot m}{R_\oplus}$. Combining the two equations and rearranging, $\frac{R_\oplus}{a} = 2 - \frac{v_2^2 R_\oplus}{GM_\odot} = 2 - x^2$. From Kepler's III Law, $\frac{T^2}{a^3} = \frac{4\pi^2}{GM_\odot}$. Thus, $T = \frac{2\pi R_\oplus^3}{\sqrt{GM_\odot}}\left(2-x^2\right)^{-3/2}$. As we can see, in order to minimise T, v_2 needs to be minimal as well.

In conclusion, I_{avg} is maximal when the satellite is launched directly against the motion of Earth. The corresponding value for I_{avg} is

$$I_{\text{avg}} = \frac{L_\odot}{4\pi R_\oplus^2}\sqrt{\frac{GM_\odot}{R_\oplus}}\frac{1}{v_2}\left(2-x^2\right)^{3/2} = \frac{L_\odot}{4\pi R_\oplus^2}\frac{(2-x^2)^{3/2}}{x},$$

where $x = \sqrt{\frac{R_\oplus}{GM_\odot}}\sqrt{v_0^2 - 2gr_\oplus} - 1$.