2. ROLLER (8 points) — *Lasse Frantti (iv,v: Jaan Kalda).(Solution of parts iv and v: Taavet Kalda).*

i) *(1 point)* If there is no friction between the cyinder and the board, the cylinder will not rotate, and we have a simple spring-block oscillator, $T_0 = 2\pi\sqrt{M/k}$.

ii) *(1 point)* Because the cylinder is not slipping, it's rotating about the point of contact with the ground. The moment of inertia with respect to the contact point is $I = Mr^2 + \frac{1}{2}Mr^2 = \frac{3}{2}Mr^2$. The angular acceleration α and acceleration a are related by $a = \alpha r$. The equation of motion therefore reads

$$I\alpha = Fr = -kxr.$$

Simplifying,

$$\frac{3}{2}Ma = -kx.$$

This corresponds to a harmonic oscillator with a period of

$$T = \frac{2\pi}{\sqrt{\frac{2}{3}\frac{k}{M}}} = 2\pi\sqrt{\frac{3}{2}\frac{M}{k}}.$$

iii) *(2 points)* The motion of the cylinder is sinusoidal:

$$x = A\sin\left(\sqrt{\frac{2}{3}\frac{k}{M}}\,t\right).$$

From the horisontal force balance,

$$-kx + F_\mu = Ma,$$

so the frictional force is linearly dependent on the acceleration and given by

$$F_\mu = -\frac{1}{2}Ma.$$

The acceleration is

$$a = \ddot{x} = -\frac{2}{3}\frac{k}{M}A\sin(\omega t).$$

Maximal frictional force is given by

$$F_\mu^{\max} = \mu Mg = \frac{1}{2}M\cdot\frac{2}{3}\frac{k}{M}A \to A < \frac{3\mu Mg}{k}.$$

Or in other words,

$$A_\star = \frac{3\mu Mg}{k}.$$

iv) *(2 points)* The equation of motion still reads

$$-kx + F_\mu = M\ddot{x},$$

but now F_μ is equal to $Mg\mu$ for most of the motion so it can be treated as a constant (the length of time where it's not equal to that gets proportionally smaller as A_0 is increased). Rewriting,

$$-k\left(x - \frac{F_\mu}{k}\right) = M\ddot{x}.$$

We see that the cylinder undergoes sinusoidal point around $x = \frac{F_\mu}{k}$ but because $A_0 \gg A_\star$, this is negligible. The rotational equation of motion yields

$$\frac{1}{2}Mr^2\alpha = rF_\mu = rMg\mu \quad \text{so} \quad \alpha = \frac{2}{Mr}F_\mu.$$

Since the direction of the frictional force is constant during $0 \le t \le T/2$, we have $\omega = \frac{2}{Mr}F_\mu t$ and this is maximal at $t = T/2$ so

$$\omega_{\max} = \frac{g\mu T}{r}.$$

v) *(2 points)* From the last part, we saw that ϵ is constant but opposite in sign for $0 < t < T/2$ and $T/2 < t < T$ with the magnitude equal to $\epsilon r = 2g\mu$. The linear acceleration, on the other hand, follows harmonical motion. From,

$$x = A_0 \cos\left(\sqrt{\frac{k}{M}}\, t\right), \quad a = \ddot{x} = -\frac{k}{M} A \cos(2\pi t/T).$$

The approximate plots are shown in the figures below.



