7. Dominoes (6 points) — *Kaarel Hänni.*

Let the i'th domino have initial angular velocity ω_i. Let us first find ω_i', the angular velocity of this domino just before colliding with the next one. A collision occurs after the domino has fallen by an angle of $\alpha = \arctan\left(\frac{1}{2}\right)$. By conservation of energy, $\frac{1}{2}I\omega_i'^2 = \frac{1}{2}I\omega_i^2 + \frac{1}{2}mg\ell(1-\cos(\alpha))$. Hence,

$$\omega_i' = \sqrt{\omega_i^2 + \frac{mg\ell}{I}(1-\cos(\alpha))} = \sqrt{\omega_i^2 + \frac{3g}{\ell}\left(1-\frac{2}{\sqrt{5}}\right)} \tag{1}$$

The collision of two dominoes is inelastic, so after the collision the two dominoes move together (for a negligibly short time before the first domino bumps into the corner of a step). After the two dominoes collide, let the angular velocity of the first one be ω_i''. Since the two dominoes move in unison,

$$\cos(\alpha)\,\ell\,\omega_i'' = \cos(\alpha)\frac{\ell}{2}\,\omega_{i+1} \implies \omega_i'' = \frac{1}{2}\omega_{i+1}. \tag{2}$$

During the collision, the force between the two dominoes is always horizontal (since there is no friction between the two), let this force be $F(t)$, where t is the time. Let us consider the angular momentum of the i'th domino w.r.t. its base.

$$I(\omega_i' - \omega_i'') = \ell\cos(\alpha)\int_{\text{beginning of collision}}^{\text{uniformiz. of velocities}} F(t)\,\mathrm{d}t \tag{3}$$

For the second domino, we have an analogous equation.

$$I\omega_{i+1} = \cos(\alpha)\frac{\ell}{2}\int_{\text{beginning of collision}}^{\text{uniformiz. of velocities}} F(t)\,\mathrm{d}t \tag{4}$$

Equations (3) and (4) together imply

$$\omega_i' - \omega_i'' = 2\omega_{i+1}. \tag{5}$$

We can now plug equations (2) and (1) into (5) to express ω_{i+1} in terms of ω_i.

$$\omega_{i+1} = \frac{2}{5}\omega_i' = \frac{2}{5}\sqrt{\omega_i^2 + \frac{3g}{\ell}\left(1-\frac{2}{\sqrt{5}}\right)} \tag{6}$$

By stability, $\omega_i = \omega_{i+1} = \omega$. This lets us finally evaluate ω.

$$\omega_i^2 = \frac{4}{25}\omega_i^2 + \frac{12g}{25\ell}\left(1-\frac{2}{\sqrt{5}}\right) \implies$$

$$\omega = \omega_i = \sqrt{\frac{4g}{7\ell}\left(1-\frac{2}{\sqrt{5}}\right)}$$