2. Tunnel diode (10 points) — *Solution by Taavet Kalda, grading schemes by Jaan Kalda, Axel Boeltzig, Bastian Hacker, and Fedor Tsybrov.*

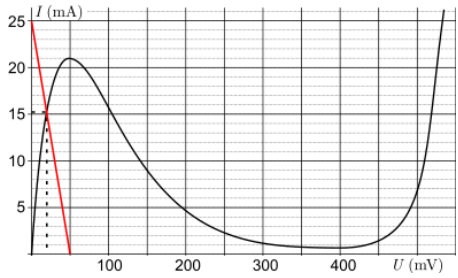
i) *(1 point)* Kirchhoff's voltage law (KVL) on the circuit:

$$\mathcal{E} = I_i r + V_i.$$

Rearranging,

$$I_i = \frac{\mathcal{E} - V_i}{r} = 25\,\mathrm{mA} - \frac{1}{2\,\Omega} V_i. \tag{2}$$

V_i and I_i also have to obey the diode's $V - I$ curve. We can find a solution graphically by plotting (2) on the $V - I$ curve. This yields $V_i = 20\,\mathrm{mV}$, $I_i = 15.3\,\mathrm{mA}$.



Grading: Writing down correct KVL — **0.3 pts**; drawing a correct line on V-I curve or explaining this procedure clearly in text — **0.3 pts** (attempts of substituting the diode with equivalent resistance, only if numerically reasonable equivalent resistance — **0.1 pts**); obtaining correct numerical value for I (from 15 to 15.5 mA) — **0.2 pts** (for I from 14 to 16 mA — **0.1 pts**); for correct numerical value for V (from 19 to 20 mV) — **0.2 pts** (for V from 18 to 22 mV — **0.1 pts**). If the pair of values is not consistent with the KVL (voltage mismatch is $\geq 1\,\mathrm{mV}$), subtract 0.1 from the voltage value subscore (if it was positive). No marks for the numerical values if obtained in a wrong way.

ii) *(1 point)* After setting $r = 0$, the KVL takes the form

$$\mathcal{E} = V_i + L\frac{\mathrm{d}I_i}{\mathrm{d}t}. \tag{3}$$

Rearranging and integrating,

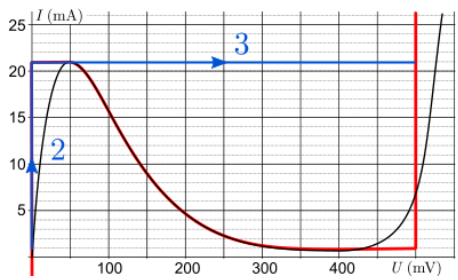
$$L\int_0^{I_1} \frac{\mathrm{d}I_i}{\mathcal{E} - V_i(I_i)} = \int_0^{t_1} \mathrm{d}t.$$

Looking at the idealised $V - I$ dependence, it's clear that $V_i(I_i) = 0$ all throughout the increase of current from $I_i = 0$ to $I_i = I_1 = 20\,\mathrm{mA}$. This simplifies the expression for t_1:

$$t_1 = \frac{L}{\mathcal{E}}\int_0^{I_1} \mathrm{d}I_i = \frac{LI_1}{\mathcal{E}} = 4 \times 10^{-8}\,\mathrm{s}.$$

Grading: Writing down correct KVL — **0.3 pts**; Integrate equation — **0.2 pts**; Note that $V_i(I_i) = 0$ — **0.2 pts**; Correct result for t_1 — **0.2 pts**.

iii) *(1 point)* Equation (3) must hold no matter what the characteristic curve for the diode looks like. This means that the current will continue to rise without any discontinuities, even if it means the voltage on the diode will jump (the inductance keeps the current from changing too fast but there is no such constraint on the voltage). The expected behaviour of $V - I$ is given in the following figure:



In leg 2 of the journey, the current increases from $I_i = 0$ to $I_i = I_2 = 21\,\text{mA}$ (measured from the figure). The time taken is $t_2 = LI_2/\mathcal{E} = 4.2 \times 10^{-8}\,\text{s}$. Since in leg 3, the change in current is 0, the time taken is essentially instantaneous compared to t_2. Hence $t_3 = 0$ for our considerations. The total time taken is then

$$t_2 + t_3 = 4.2 \times 10^{-8}\,\text{s}.$$

Grading: Description / understanding of the processes — **0.5 pts**; Calculation t_2 — **0.2 pts**; Result $t_2 + t_3$ — **0.3 pts**.

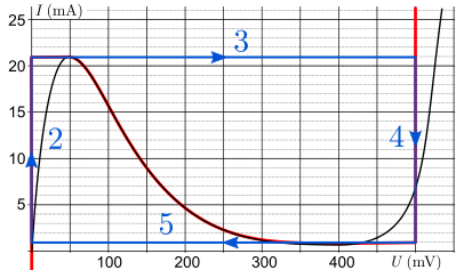
iv) *(2 points)* We can use similar logic as before to deduce how the voltage and current behave as a function of time. Since the equilibrium voltage $\mathcal{E} = 250\,\text{mV}$ lies between the two peaks in the $V - I$ curve, the current will perform a horizontal jump as before. At $V_2 = 500\,\text{mV}$, equation (3) takes the form

$$\mathcal{E} = V_2 + L\frac{\mathrm{d}I_i}{\mathrm{d}t},$$

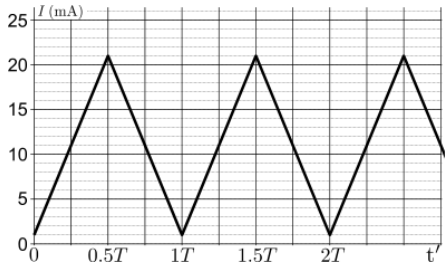
so

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathcal{E} - V_2}{L} < 0.$$

Hence, I_i will continue to decrease from I_2 to $I_3 = 1\,\text{mA}$. Like before, the voltage will then instantaneously jump from V_2 to 0 and the cycle starts again. A sketch of a single cycle is shown in the following figure.



The time taken in legs 3 and 5 are effectively 0 and because the deviation of the voltage from $\mathcal{E}$ in legs 2 and 4 is the same, alongside with the change in current, the time duration for 2 and 4 must also be the same. The change in current is $I_2 - I_3 = 20\,\text{mA} = I_1$. Hence $t_2 = t_4 = t_1$ and the duration of one full period is $T = t_2 + t_3 + t_4 + t_5 = 2t_1 = 8 \times 10^{-8}\,\text{s}$. A sketch of I as a function of time is shown in the following figure. t' has the moment when the current is at its minimum at $t' = 0$.



Grading: Writing down correct KVL at V_2 — **0.3 pts**; Argument that Δt_3 and $\Delta t_5 = 0$ — **0.3 pts**; Calculation Δt_4 — **0.3 pts**; Period of oscillation — **0.3 pts**; Amplitude of oscillation — **0.3 pts**; Offset of oscillation — **0.2 pts**; Correct plot, starting from $I = 0$ — **0.3 pts**.

v) *(2 points)* The system operates in 4 distinct modes as the battery voltage is varied:

1. Applied voltage is smaller than the first peak in the $V - I$ curve. In that case, the current will increase from 0 to $I = 21\,\text{mA}$ and reach the equilibrium position at $V = \mathcal{E}$. Indeed, it's an equilibrium because it satisfies KVL given by (3):

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 0 = \frac{\mathcal{E} - V}{L}.$$

Hence, the ammeter measures a constant 21 mA.

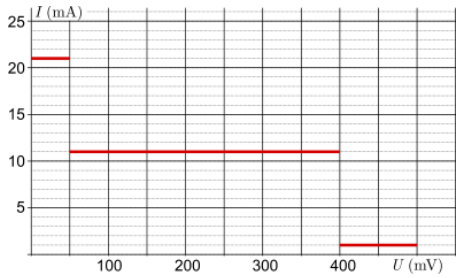
2. Applied voltage is between the two peaks in the $V-I$ curve. The system will follow a similar trajectory to the one exhibited in iv) since the same argumentation holds. Following the same notation as in iv), the average current in leg 2 is the arithmetic average between 1 mA and 21 mA (because the current is increasing at a constant rate). Leg 2 thus has an average current of 11 mA. Leg 4 similarly has the same average current. Leg 3 and 5 don't contribute to the average current because they happen effectively instantaneously. The total average current is then 11 mA.

3. Applied voltage is bigger than the second peak in the $V-I$ curve but smaller than 500 mV. In the beginning, the current will increase to 21 mA and make a horizontal jump, just as expected. Then the current will decrease to 1 mA and the voltage takes the value of the battery and reaches an equilibrium without performing any additional jumps. The average current is thus 1 mA.

4. If the applied voltage is bigger than 500 mV, then the current will blow up to infinity (in our ideal model) and that's not physical. Hence, the current is undefined.

Grading: For each of the four modes, Identification — **0.2 pts**; Determination of constant value — **0.3 pts**.

From the $V-I$ curve, the first peak has a voltage of $V_3 = 50\,\text{mV}$, second a voltage of $V_4 = 400\,\text{mV}$. The four scenarios can be summarised in the following plot:



vi) *(1 point)* First, we'll find the operational mode using the same graphical method as in part i). The graphed line has an equation of

$$I_i = \frac{\mathcal{E} - V_i}{r} = 75\,\text{mA} - \frac{1}{2\,\Omega} V_i,$$

shown in red in the figure. The steady voltage and current are measured to be $V_0 = 125\,\text{mV}$ and $I_0 = 11.9\,\text{mA}$. For small perturbations from the steady state, we can use Taylor series while neglecting higher orders:

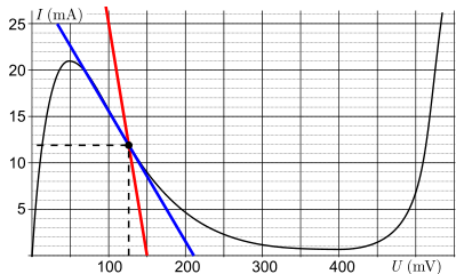
$$V_0 + \delta V(t) = V(I_0 + \delta I(t)) \simeq V_0 + \delta I(t) \left.\frac{\mathrm{d}V}{\mathrm{d}I}\right|_{I_0}.$$

Therefore

$$\delta V = \left.\frac{\mathrm{d}V}{\mathrm{d}I}\right|_{I_0} \delta I = R_d\,\delta I.$$

We can express $\left.\frac{\mathrm{d}V}{\mathrm{d}I}\right|_{I_0}$ graphically by drawing a line tangent to the $V-I$ curve going through the steady state. The derivative is then found by dividing the horizontal projection with the vertical, while keeping track of the sign:

$$R_d = \left.\frac{\mathrm{d}V}{\mathrm{d}I}\right|_{I_0} = \frac{178\,\text{mV}}{-25\,\text{mA}} = -7.1\,\Omega.$$



Grading: Writing down correct KVL — **0.1 pts**; drawing a correct line on $V-I$ curve or explaining this procedure clearly in text — **0.2 pts**; obtaining correct numerical value for I (from 11 to 13 mA) — **0.1 pts**; for correct numerical value for V (from 115 to 135 mV) — **0.1 pts**; for drawing tangent line to the curve through the intersection point — **0.2 pts**; determining correctly R_d as the slope of the tangent (from -6.5 to -7.6) — **0.3 pts**; if the result is from -6 to -8 — **0.2 pts**, if it is from -5 to -9 — **0.1 pts**. Zero marks if the minus sign is missing. If final result is correct, but the values of I_0 and V_0 not shown, no penalty is applied.

vii) *(2 points)* In order to find the stability condition, one could operate with complex impedances and write down the resonance condition

$$r + i\omega L + \frac{R_d}{i\omega R_d C + 1} = 0,$$

hence, denoting $\lambda = i\omega$,

$$(r + \lambda L)(\lambda R_d C + 1) + R_d = 0.$$

A more tedious but perhaps clearer way would be to write down the KVL and solve the resulting differential equation. Let the deviation of the charge on the capacitor from steady state be δq. Then, from KVL, $\delta I R_d = \delta q/C$. Hence, the current through the resistor r and inductor is $\delta I + \dot{\delta q}$, where $\dot{\delta q} = \dot{\delta I} R_d C$. KVL for the whole circuit takes the form

$$\begin{aligned} 0 &= (\delta I + \dot{\delta q})r + L\frac{\mathrm{d}}{\mathrm{d}t}(\delta I + \dot{\delta q}) + \delta I R_d \\ &= R_d LC\ddot{\delta I} + (L + R_d rC)\dot{\delta I} + (R_d + r)\delta I \\ &= \ddot{\delta I} + \left(\frac{1}{R_d C} + \frac{r}{L}\right)\dot{\delta I} + \frac{r + R_d}{R_d LC} \\ &= \ddot{\delta I} + b\dot{\delta I} + c, \end{aligned}$$

where $b = \left(\frac{1}{R_d C} + \frac{r}{L}\right)$, $c = \frac{r+R_d}{R_d LC}$. This is a second order differential equation. Depending on the values for b and c, the solution might grow exponentially. The standard method for solving this type of equation involves making an educated guess and plugging it into the equation. In this case, an exponential solution of the form $\delta I = \delta I_0 \exp(\lambda t)$ will work. Note that this is all equivalent to operating with complex impedances but with $\lambda = i\omega$. Substituting the ansatz into the differential equation and reducing the prefactors, one gets the characteristic equation:

$$\lambda^2 + b\lambda + c = 0.$$

This is a quadratic equation with two solutions

$$\lambda_{12} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c}.$$

λ_{12} can be either both real or both complex, depending on the sign of the discriminant. If $\lambda_j = m_j + n_j i$, where m, and n are both real, then

$$\delta I = \sum_{j=1}^{2} \delta I_{0j} e^{m_j t} \left(\cos(n_j t) + i \sin(n_j t)\right).$$

It can be seen that for the solution to be stable, $m < 0$ is needed as that leads to an exponential decay in the current. In other words, the real part of λ has to always be negative, otherwise the current will start growing exponentially. With careful analysis, it's possible to determine necessary conditions for b and c for this to be the case.

Vieta's second formula states that $\lambda_1 \lambda_2 = c$. If λ is real, then this means that c has to be positive, because otherwise either λ_1 or λ_2 is negative. If λ is complex, then λ_2 and λ_1 are each-other's complex conjugates and so their product must be positive. Hence, $c > 0$ regardless of whether λ is real or complex.

According to Vieta's first formula, $\lambda_1 + \lambda_2 = -b$. If λ is real, then their sum has to be negative, otherwise at least one of λ_1 and λ_2 is positive. Hence, $b > 0$. If λ is complex, then their sum is purely real (because they're each-other's complex conjugates) and hence again, the sum has to be negative for the real parts to be negative. Hence, $b > 0$ must always hold.

The $b > 0$ and $c > 0$ are necessary and sufficient conditions for the solution to be stable. Condition $b > 0$ implies

$$\left(\frac{1}{R_d C} + \frac{r}{L}\right) > 0$$

so

$$L < |R_d| r C = 4.3 \times 10^{-10}\,\mathrm{H} = 0.43\,\mathrm{nH}.$$

Inequality $c > 0$ implies

$$\frac{r + R_d}{R_d L C} > 0,$$

hence

$$r + R_d < 0.$$

As can be seen, the value for L can't exceed 0.43 nH.

Grading: consideration of current small deviation — **0.1 pts**; relationship between capacitor charge δq and diod current δI — **0.1 pts**; initial KVL for whole circuit — **0.3 pts**; correct differential equation — **0.3 pts**; quadratic equation — **0.2 pts**; analyze of quadratic equation according to the problem — **0.6 pts**; expression for inductance: $L < |R_d| r C$ — **0.3 pts**; numerical answer: $L < 0.43\,\mathrm{nH}$ — **0.1 pts**.