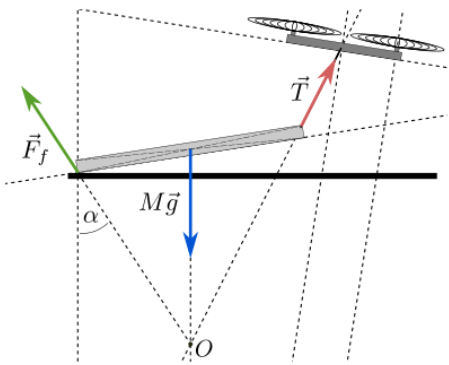
4. Drone (9 points) — *Solution by Taavet Kalda, grading schemes by Oleg Košik, Jānis Cimurs, and Joonas Kalda.*

i) *(2 points)* Let the mass of the cuboid be M. There are three forces acting on the drone: the resultant of friction and the normal force $\vec{F}_f$, rope tension $\vec{T}$ directed along the rope, and gravitational acceleration $M\vec{g}$ directed vertically down from the centre of the cuboid. Since the cuboid is sliding with constant speed, the three forces must balance each other out. The only way for this to be possible is if the vectorial extensions of the forces intersect in one point, O.

One can prove this by contradiction. If the forces don't intersect in a single point, one needs only consider the torque around one of the intersection points to see that there is non-zero torque and that the forces aren't in equilibrium.

If the normal force is N, then the frictional force is $N\mu$ so the resultant $\vec{F}_f = N\hat{y} - N\mu\hat{x}$. Therefore, $\vec{F}_f$ is always directed at an angle $\alpha = \arctan\mu$ with respect to the vertical.

Since the starting point and direction of the forces of gravity and tension are known, one can reconstruct the position of O and $\vec{F}_f$. Because $\mu = \tan\alpha$, one can conveniently measure μ as the ratio of the horizontal and vertical projection of $\vec{F}_f$: $\mu \approx 0.659$.

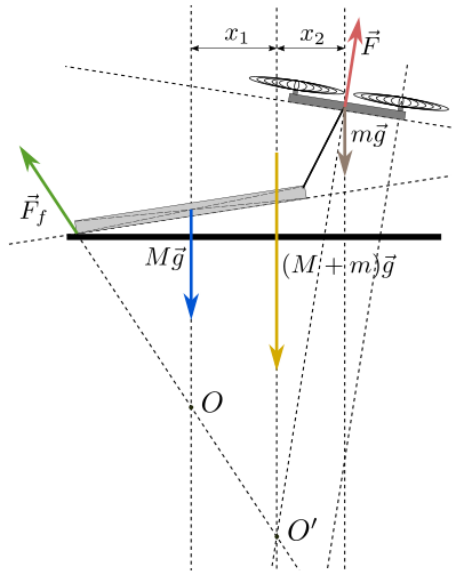


ii) *(2 points)* Consider the system made up of the cuboid and the drone. Once again, there are three forces acting on this system: gravitational force $(M+m)\vec{g}$, friction $\vec{F}_f$, and the force $\vec{F}$ keeping drone afloat. The thrust for the drone is directed along the symmetry axis of the drone. Since the forces are in equilibrium, their extensions must intersect in one point O'. Owing to the last part, O' can be found by intersecting the frictional force and the thrusting force. Since gravitational force is vertical, we can find the horizontal projection of the centre of mass. If x_1 and x_2 are the horizontal distances from O' to the centres of the cuboid and drone respectively, then

$$\frac{x_1}{x_2} = \frac{M}{m}.$$

From the figure we measure $x_1/x_2 = 0.796$ and so

$$M = 0.796m = 0.796\,\mathrm{kg}.$$



Grading for i) and ii)

Solutions that use force balance and torque balance in i) and force balance in ii):

i) correctly identifying all forces acting on cuboid — **0.2 pts**; use that $\mu = \frac{F_f}{N}$, where F_f is friction force and N is normal force — **0.2 pts**; writing force balance equations using angles — **0.4 pts**; writing torque balance equation — **0.4 pts**; deriving μ — **0.4 pts**; numerical result with high enough precision — **0.4 pts**; (error within 5% — 0.4pts, error within 10% — 0.2pts)

ii) correctly identifying all forces acting on drone — **0.2 pts**; writing force balance equations using angles — **0.6 pts**; combining with equations form part i) and deriving M — **0.8 pts**; numerical result with high enough precision — **0.4 pts**; (error within 5% — 0.4pts, error within 10% — 0.2pts)

Remark. Solutions that assume that cuboid is linear, get 0 for precision for both parts i) and ii), but there are no deductions for deriving μ and M.

*Solutions that use point O in **i)**:* correctly identifying all forces acting on cuboid — **0.2 pts**; use fact that vectorial extensions intersect at one point or another way to take into account torque balance for point O — **0.8 pts**; use that $\mu = \tan\alpha$ or $\mu = \frac{F_f}{N}$, where F_f is friction force and N is normal force — **0.2 pts**; deriving μ — **0.4 pts**; numerical result with high enough precision — **0.4 pts**.

*Solutions that use point O' in **ii)**:* correctly identifying all forces acting on system — **0.2 pts**; use fact that vectorial extensions intersect at one point or another way to take into account torque balance for point O' — **0.8 pts**; use torque balance for gravitational forces — **0.4 pts**; express formula for mass M — **0.2 pts**; numerical result with high enough precision — **0.4 pts**.

iii) *(2 points)* Imagine a pocket of air with fixed mass moving around in the atmosphere. Let the pocket's volume be $V = V(z)$. In an adiabatic atmosphere, $pV^\gamma = \text{const}$, where $\gamma = c_p/c_v = 1.39$. Now, $pV \propto T$ and $\rho \propto V^{-1}$, so

$$pV^\gamma \propto V^{\gamma-1}T \propto \rho^{1-\gamma}T = \text{const}.$$

Hence,

$$\rho(z) = \rho_0 \left(\frac{T(z)}{T(0)}\right)^{\frac{1}{\gamma-1}} = \rho_0 \left(1 - \frac{gz}{c_p T_0}\right)^{\frac{1}{\gamma-1}}.$$

Grading: There are two expected solutions. One of them is given by the sample solution while the other involves integrating $\mathrm{d}\rho$ from $z = 0$ to z.

Grading for sample solution:

Using or deriving the adiabatic relation $pV^\gamma = \text{const}$ — **0.6 pts**; Using or deriving an expression for $\gamma = c_p/c_v$ — **0.2 pts**; Deriving an exact expression for ρ, or obtaining its dependence on V and/or on p, T — **0.6 pts**; Obtaining the correct expression for ρ — **0.6 pts**;

Grading for alternative solution:

Using or deriving the relation for the pressure change $\mathrm{d}p(z) = -\rho(z)g\,\mathrm{d}z$ — **0.1 pts**; Using the relation $c_p - c_v = R/\mu$ — **0.2 pts**; Using ideal gas law or equivalent to get another differential — **0.3 pts**; Obtaining an expression for ρ in terms of other quantities of interest — **0.6 pts**; Correctly setting up the integral for ρ and z or equivalent quantities — **0.2 pts**; Obtaining the correct expression for ρ — **0.6 pts**;

iv) *(3 points)* The drone stays afloat by using the motor to push air through its propellers. The amount of thrust is clearly a function of the density of the air and the speed v at which air goes through the propellers.

Force balance can be written down as $F - m_{\text{tot}}g = 0$, where F is the vertical thrust and m_{tot} the total mass of the drone. If A is the effective area of the propellers, it's possible to write down the expression for F by either using the dynamical pressure ρv^2 or by considering the conservation of momentum. In a time interval Δt, a volume of $\Delta V = Av\Delta t$ of air passes through the propellers. The air volume carries momentum $\Delta p = \Delta V \rho v$, so the thrust is given by $F = \Delta p/\Delta t = A\rho v^2$.

Secondly, it's possible to tie the power output P of the motor with outside air density and speed. Notably, the air is pushing the propellers vertically up with a force F. In order to function, the propeller blades need to be slanted. This amounts to a torque that's proportional to F. Further, it's clear that the rotational speed of the propeller blades is also proportional to v. This means that the output power of the motor is proportional to the product of F and v and so $P \propto \rho v^3$. In our considerations, the output power of the drone is fixed so $v \propto \rho^{-1/3}$ and $F \propto \rho(\rho^{-1/3})^2 = \rho^{1/3}$. From force balance, $F = m_{\text{tot}}g$. Hence, $m_{\text{tot}} \propto \rho^{1/3}$. Evaluating the ratio at $z = 0$ and $z = z_{\max}$, one gets

$$\frac{1.5m}{m} = \left(\frac{\rho(0)}{\rho(z_{\max})}\right)^{1/3} = \left(1 - \frac{gz_{\max}}{c_p T_0}\right)^{-\frac{1}{3(\gamma-1)}},$$

and so

$$z_{\max} = \frac{c_p T_0}{g}\left(1 - 1.5^{-3(\gamma-1)}\right) = 11.3\,\text{km}.$$

Grading: Writing down the force balance equation — **0.4 pts**; Deriving a relation between the thrust and the air density and speed by either considering momentum conservation over a small time interval or using the expression for dynamical pressure — **0.8 pts**; Tying the motor power with air density and speed — **0.6 pts**; Finding a relation between the maximum lift power and air density — **0.4 pts**; Evaluating the two conditions for maximum lift power of the drone at $z = 0$ and $z = z_{\max}$ — **0.2 pts**; Obtaining the correct expression for $z_{\max}$ — **0.4 pts**; Obtaining the correct numerical value for $z_{\max}$ — **0.2 pts**.