5. Bottle's sound (8 points) — *Solution by Jaan Kalda, marking schemes by Eero Uustalu (task i), Topi Löytäinen, and Miha Marttinen (tasks ii, iii).*

i) *(4 points)* The following frequencies can be obtained for 1-litre bottle, measured frequency of sound is tabulated versus the volume of water in the bottle.

V (ml)	0	100	200	310	400	
f (Hz)	144	151	163	175	185	
V (ml)	500	600	700	800	880	930
f (Hz)	205	230	260	325	420	520

Grading: The measurement data give evidence that volumes have been measured correctly: **0.2 pts** (for instance, if a portion of water was added without making a notice of it, all the subsequent volumes are offset by a certain amount, and in that case, this 0.2 pts is not awarded)
There is at least one measurement with empty bottle ($V = 0$) **0.2 pts**.
There is at least one measurement with less than 10% of the bottle's volume being empty **0.2 pts**.
There is at least one measurement in each of the volume ranges: $0 < V/V_0 \leq 20\%$; $20 < V/V_0 \leq 40\%$; $40 < V/V_0 \leq 60\%$; $60 < V/V_0 \leq 70\%$; $70 < V/V_0 \leq 80\%$; $80 < V/V_0 \leq 90\%$; **0.2 pts**.
Quality of measurements: in f^{-2} versus V graph, the data should lie on a strait line. Every point (up to 10th point) which is "good", i.e. lies on a line — **0.2 pts**. If an outlier point corresponds to the second harmonic, 0.1 pts is given instead of 0.2 pts.
Volume of the bottle measured: **0.2 pts**. If volume is not measured but read from the label — **0.1 pts**.
If instead of the volume of water, the volume of air is used, the total score for task i is multiplied by 0.8 and rounded up to the first decimal digit. The same applies if frequency is not recorded in Herz, but musical notes.
If only a graph is built with no tabulated data, subtract 10% from the final result of this subtask.

ii) *(1.5 points)* We can consider the air in the region of the bottle's neck of volume $v \ll V_0 = 1\,\mathrm{l}$ as a mass $m = \rho_a v$ (ρ_a denotes the density of air) which can move back and forth while the air inside the bulk of the bottle serves as a spring. If the air inside the neck moves by distance x, the volume inside the bottle is changed by Ax, where A denotes the cross-section area of the neck. The process is fast, characteristic time is around few milliseconds, so we can consider it to be adiabatic (characteristic time of thermalization is on the order of a second). From $pW^\gamma = \text{const}$ (where $W = V_0 - V$ denotes the air volume inside the bottle) we obtain $\ln p + \gamma \ln W = \text{const}$, hence $\frac{\Delta p}{p} + \gamma \frac{\Delta W}{W} = 0$, i.e.

$$\Delta p = -\gamma p \frac{\Delta W}{W} = \gamma p \frac{Ax}{W}.$$

Now we can write the equation of motion for the air inside the neck as

$$\rho_a v \ddot{x} = -\Delta p A = -x\gamma p \frac{A^2}{W},$$

hence the frequency

$$f = \frac{1}{2\pi}\sqrt{\gamma \frac{pA^2}{\rho_a v W}} = \frac{1}{2\pi}\sqrt{\gamma \frac{RT}{\mu}\frac{A^2}{v(V_0 - V)}}.$$

Grading: 1.5p: If $f \propto 1/\sqrt{V_0 - V}$ [or $f \propto (V_0 - V)^{-n}$ with $n \approx 0.5$] either based on data analysis or adiabatic oscillation approach.
0.5p: Data analysis leading to unphysical (linear, quadratic, exponential,...) dependence.
1p: Standing wave approach or data analysis leading to $1/V$ dependence.

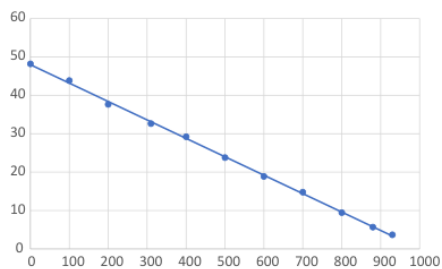
iii) *(3 points)* Based on our previous result, we can see that the squared period

$$T^2 = 4\pi^2 \frac{\mu}{RT} \frac{v(V_0 - V)}{A^2}$$

is a linear function of the volume of water. Using the measurement data we calculate the squared period (ms^2).

V (ml)	0	100	200	310	400	
T^2 (ms^2)	48.2	43.9	37.6	32.7	29.2	
V (ml)	500	600	700	800	880	930
T^2 (ms^2)	23.8	18.9	14.8	9.5	5.7	3.7

These data are plotted below.



The linear fit of these data yields

$$T^2 = 48\,\mathrm{ms}^2 - V \cdot 48\,\mathrm{ms}^2/\mathrm{l},$$

so that

$$f = \left(48\,\mathrm{ms}^2 - V \cdot 48\,\mathrm{ms}^2/\mathrm{l}\right)^{-1/2}.$$

Grading: 1p: For graph (labels, units)
1p: Linearization or comparison to model prediction.
1p: For parameterization consideration either theoretical or physical (heuristic) justification.