1. Photon rocket (5 points) — *Solution by Taavet Kalda, grading schemes by Oskar Vallhagen and Konstantin Dukach.*

i) *(1 point)* At non-relativistic speeds, we can apply classical momentum and energy conservation to find the acceleration in terms of the antimatter burning rate μ. In a time interval Δt, a mass of $\Delta m = \mu \Delta t$ antimatter annihilates with an equal mass of matter. The resulting photons have an energy equal to the annihilated rest energy $\Delta E = 2\Delta m c^2$. For maximal acceleration, the photons have to be all emitted in the same direction (can be achieved, for example, using mirrors). The resulting photon cloud will then have a momentum of $\Delta p = \Delta E/c$. From the conservation of momentum, the space ship must get a momentum boost in the opposite direction, equal to $\Delta p = M\Delta v = Mg\Delta t$. Combining everything, we get $\mu = Mg/(2c) = 1.635 \times 10^{-3}\,\mathrm{kg/s}$.

Grading:

- Introducing time Δt and expressing $\Delta m = \mu \delta t$ **0.2 pts**
- Photon energy $\Delta E = 2\Delta m c^2$ **0.2 pts**
- Photon momentum $\Delta p = \Delta E/c$ **0.2 pts**
- Rocket momentum $\Delta p = Mg\Delta t$ (half point if only force $F = Mg$ is stated) **0.2 pts**
- Final answer **0.2 pts**
- Missing factor 2 accounting for both antimatter and matter fom interstellar plasma **−0.2 pts**
- Finding photon momentum by reasoning that the annihilated mass can be regarded as expelled with the speed of light **−0.2 pts**

ii) *(3 points)* The final speed is easiest to find by applying energy and momentum (4-momentum) conservation in the initial and final configurations. If the final rest mass of the ship is m_f, then $M - m_f$ of antimatter is burnt throughout the acceleration and an equal amount of matter is burnt from the interstellar space. Hence, it makes sense to consider the system of the ship + the burnt interstellar gas.

The initial rest mass of the system is $M+(M-m_f) = 2M-m_f = M_i$. This corresponds to an energy of $E_i = M_i c^2$ and momentum $p_i = 0$ due to the gas and the ship initially being at rest.

In the final configuration, we have the space ship moving with a speed of v with energy E_f and momentum p_f. We also have a photon cloud with energy E_p and momentum p_p, moving opposite to the direction of the space ship. Then the energy and momentum conservation simply read as $E_f + E_p = E_i$ and $p_i = 0 = p_f + p_p$. We also have the 4-momentum invariant for both the space ship and the photon gas. They read $E_f^2 - p_f^2 c^2 = m_f^2 c^4$ and $E_p^2 - p_p^2 c^2 = 0$. Solving the 4 equations (with 4 unknowns), we get

$$p_f = \frac{M_i^2 - m_f^2}{2M_i} c, \qquad E_f = \frac{M_i^2 + m_f^2}{2M_i} c.$$

We could find the velocity by solving for v_f in the expression for the relativistic energy $E_f = m_f \gamma_f c^2$ with $\gamma_f = (1 - v_f^2/c^2)^{-0.5}$. A faster way, however, would be to use the expression for final momentum, $p_f = m_f \gamma_f v$, and notice that

$$\begin{aligned} v_f = \frac{p_f}{E_f} c^2 &= \frac{M_i^2 - m_f^2}{M_i^2 + m_f^2} c \\ &= \frac{(2M - m_f)^2 - m_f^2}{(2M - m_f)^2 + m_f^2} c \\ &= \frac{180}{181} c \approx 0.9945c. \end{aligned}$$

Grading:

- Realising that the problem can be solved by considering conservation of momentum and energy between initial and final state only **0.5 pts**
- Correct conservation laws for momentum and energy (half of the points for each) **0.5 pts**
- Relating energy and momentum with 4-momentum invariants (or equivalent, half of the points each for rocket and photons) **1 pts**
- Solving the equation system in terms of energies and momentum **0.5 pts**
- Solving for v_f **0.5 pts**

Non-relativistic solution:

- Realising that the problem can be solved by considering conservation of momentum and energy between initial and final state only **0.25 pts**
- Correct conservation laws for momentum and energy (half of the points for each) **0.25 pts**
- Relating energy and momentum to v_f **0.25 pts**
- Final answer (if commenting on unphysical results, otherwise no points) **0.25 pts**

Solution 2 by Oskar Vallhagen

An alternative way to solve the problem is to find an expression for the reactive force felt by the rocket, and use it to derive an equation for the resulting velocity change. During a time $\mathrm{d}t$, the energy released as photons from the burnt rocket antimatter is $\gamma\mu\,\mathrm{d}t\,c^2$ (including the kinetic energy of this mass) and the energy released from the ordinary matter needed for the annihilation is $\mu\,\mathrm{d}t\,c^2$. As the photon momentum scales as the inverse of the wavelength, which is red-shifted by a factor of $\sqrt{1+v/c}/\sqrt{1-v/c}$ in the rest frame compared to the rocket frame, the corresponding change in photon momentum in the rest frame is

$$\mathrm{d}p_{ph} = (\gamma+1)\mu\,\mathrm{d}t\,c\sqrt{\frac{1-v/c}{1+v/c}}.$$

The force acting on the rocket can now be expressed as

$$\begin{aligned}\frac{\mathrm{d}p}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}v}(\gamma v)\,m\frac{\mathrm{d}v}{\mathrm{d}t} + \gamma v\frac{\mathrm{d}m}{\mathrm{d}t}\\ &= m\left(\gamma^3\frac{v^2}{c^2}+\gamma\right)\frac{\mathrm{d}v}{\mathrm{d}t} - \gamma v\mu\\ &= m\gamma^3\frac{\mathrm{d}v}{\mathrm{d}t} - \gamma v\mu = \frac{\mathrm{d}p_{ph}}{\mathrm{d}t}\\ &= (\gamma+1)\mu c\sqrt{\frac{1-v/c}{1+v/c}}.\end{aligned}$$

We now change variable from time to mass by noting that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}m}\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{\mathrm{d}v}{\mathrm{d}m}\,\mu,$$

yielding

$$-m\gamma^3\frac{\mathrm{d}v}{\mathrm{d}m} - \gamma v = (\gamma+1)c\sqrt{\frac{1-v/c}{1+v/c}}.$$

This equation can be solved by separation of variables according to

$$\int_M^{m_f} \frac{\mathrm{d}m}{m} = \int_0^{v_f} -\frac{\gamma^3}{\gamma v + (\gamma+1)c\sqrt{\frac{1-v/c}{1+v/c}}}\,\mathrm{d}v,$$

which can be simplified to

$$\ln\left(\frac{m_f}{M}\right) = \int_0^{v_f/c} -\frac{\mathrm{d}x}{(1-x)\left(\sqrt{1-x^2}+x+1\right)} = I(v_f/c) - I(0),$$

with $x = v/c$ and I is the indefinite integral. To evaluate this integral, we first make the substitution $x = \sin y$. This gives $\sqrt{1-x^2} = \cos y$ and $\mathrm{d}x = \cos y\,\mathrm{d}y$, so that the (indefinite) integral becomes

$$I = \int -\frac{\cos y\,\mathrm{d}y}{(1-\sin y)(\cos y + \sin y + 1)}.$$

We now make the further substitution $z = \tan y/2$, which gives

$$\sin y = \frac{2z}{1+z^2}, \qquad \cos y = \frac{1-z^2}{1+z^2}, \qquad \mathrm{d}y = \frac{2}{1+z^2}\,\mathrm{d}z,$$

so that the integral becomes, after some manipulation,

$$I = \int -\frac{\mathrm{d}z}{1-z} = \ln(1-z).$$

Substituting back, we use the trigonometric expressions $\sin y = 2\sin(y/2)\cos(y/2)$ and $\cos y = 2\cos^2(y/2) - 1$ to rewrite $z = \tan(y/2) = \sin(y/2)/\cos(y/2) = \sin y/(2\cos^2(y/2)) = \sin y/(1+\cos y) = x/(1+\sqrt{1-x^2})$, which after insertion into the original integral gives

$$\begin{aligned}
\ln\left(\frac{m_f}{M}\right) &= \ln\left(1 - \frac{x_f}{1+\sqrt{1-x_f^2}}\right) \\
\Rightarrow \frac{1}{10} &= 1 - \frac{x_f}{1+\sqrt{1-x_f^2}} \\
\Rightarrow x_f &= \frac{180}{181}.
\end{aligned}$$

Grading:

- Correct photon momentum change in the rest frame **0.5 pts**
- Expanding the momentum change of the rocket in terms of v and setting up total momentum (or force) balance **0.25 pts**
- Eliminate the time variable **0.25 pts**
- Express solution as an integral using separation of variables **0.5 pts**
- Simplifying the integral and first substitution **0.5 pts**
- Second substitution **0.5 pts**
- Substitute back and calculate final answer **0.5 pts**

Non-relativistic solution:

- Momentum balance **0.25 pts**
- Eliminate time variable and express as integral using separation of variables **0.25 pts**

- Evaluate the integral **0.25 pts**
- Final answer (if commenting on unphysical results, otherwise no points) **0.25 pts**

iii) *(1 point)* The last photon is emitted when the space ship moves at a speed of v_f. The photon is observed by a stationary observer. We can directly use the relativistic Doppler shift effect to find

$$f_{\text{obs}} = f_0\sqrt{\frac{1-v/c}{1+v/c}} = \frac{m_f}{M_i}f_0 = \frac{m_f}{2M-m_f}f_0 = \frac{1}{19}f_0.$$

Grading:

- Correct relativistic expression for doppler shift (-0.3 pts if non-relativistic or only valid for $v_f \ll c$) **0.5 pts**
- Final answer (no points if unphysical, eg negative frequencies, even if consistent with previous calculations) **0.5 pts**