2. Gas and fluid flows (10 points) — *Solution by Taavet Kalda, grading schemes by Joonas Kalda, Mihkel Kree, Andres Põldaru, Oleg Košik.*

i) *(1 point)* As the plate falls, it will rotate around the bump without slipping and push the air out from beneath it, at ever faster speeds, the closer it gets to the bottom plate. As such, part of the rotational energy of the plate is transferred over to the escaping air molecules. Further, the pressure and temperature of the gas is uniform, because of the incompressibility condition.

Since the problem is 2-dimensional, the mass, volumes, moment of inertia and other quantities are per unit length of the system (on the figure, into the page). Let x mark the distance from the bump and v denote the horizontal speed of air at x. There is a volume of air equal to $V(x) = xhx/(2L)$ between $x = 0$ and x. As the plate falls down, $V(x)$ gets smaller and as a result, air is pushed out. Consider a small time interval $\mathrm{d}t$. In that time interval, h changes by $\dot{h}\,\mathrm{d}t = -\omega L\,\mathrm{d}t$. From the conservation of air particles, $0 = \mathrm{d}V(x) + v(x)hx/L$ with $\mathrm{d}V(x) = -x^2\omega/2$. Hence,

$$v(x) = \frac{xL\omega}{2h}.$$

Evaluated at $x = L$, this yields

$$v(x = L) = \frac{L^2\omega}{2h}.$$

Grading:

- Stating the idea of using conservation of mass (or implicitly using it) to find the velocity **0.4 pts**
- Equation for the conservation of mass **0.4 pts**
- Correct final expression **0.2 pts**

ii) *(2.5 points)* Since the air flow is laminar and there is no diffusion, all of the lost rotational energy from the falling glass plate will be converted into kinetic energy of air particles. As such, we have $K_{\mathrm{rot}} + K_{\mathrm{air}} = \mathrm{const}$. We found from the previous parts that the air is pushed out at ever faster speeds (between the plates, $v(x) \gg L\omega$) from between the two plates. However, outside of the two plates, the flow will diffuse fast in all directions, such that the vast majority of kinetic energy will be concentrated between the two plates.

The kinetic energy of the air between the two plates is

$$\begin{aligned} K_{\mathrm{in}} &= \int_0^L \mathrm{d}x\,\frac{hx}{L}\,\rho_a\,\frac{1}{2}v(x)^2 \\ &= \frac{\rho_a L\omega^2}{8h}\int_0^L x^3\,\mathrm{d}x \\ &= \frac{\rho_a L^5\omega^2}{32h}, \end{aligned}$$

and the rotational energy of the glass plate is

$$K_{\mathrm{rot}} = \frac{I\omega^2}{2} = \frac{m_{\mathrm{glass}}L^2}{3}\,\frac{\omega^2}{2} = \frac{L^3 t\rho_g\omega^2}{6}.$$

Energy conservation then reads as

$$\begin{aligned} \frac{\rho_a L^5\omega^2}{32h} + \frac{L^3 t\rho_g\omega^2}{6} &= \mathrm{const} \\ &= \frac{\rho_a L^5\omega_0^2}{32h_0} + \frac{L^3 t\rho_g\omega_0^2}{6}, \end{aligned}$$

and hence,

$$\omega = \omega_0 \sqrt{\frac{1 + \frac{3\rho_a L^2}{16\rho_g t}\frac{1}{h_0}}{1 + \frac{3\rho_a L^2}{16\rho_g t}\frac{1}{h}}}.$$

We can see that $\lim_{h\to 0} \omega = 0$, i.e. the air acts as a cushion and stops the glass plate before it hits the stationary plate.

Grading:

- Noticing from the laminarity of the flow that the kinetic energy of the glass plate and its surrounding air is conserved. If the flow is assumed to be dissipative, and the problem is otherwise solved correctly, the maximum score is reduced by 0.3 pts from this part. **0.6 pts**
- Calculating the kinetic energy of the air, of which: **1.2 pts**
  - Noting that the majority of the air's kinetic energy is between the two glass plates (no formal proof required) **0.6 pts**
  - Finding the air velocity at a distance x from the pivot **0.2 pts**
  - Correct setup for the integral for the kinetic energy **0.3 pts**
  - Correct final expression for the kinetic energy of air **0.1 pts**
- Calculating the kinetic energy of the glass slab, of which: **0.4 pts**
  - Correct expression for the moment of inertia **0.2 pts**
- Final expression for angular speed as a function of h **0.3 pts**

iii) *(3 points)* The incoming water from '2' will spread out axisymmetrically along the space between the stone disk and the ceiling. After that, it will spread into the basin and eventually leave through the outgoing pipe '3'. The stone disk is kept up by the pressure differences between the top and bottom of the disc, arising from the flow speed of the water differing on either sides. The fact that pressures differ on boths sides can be seen directly by the application of the Bernoulli principle $p + \rho_w v^2/2 = \text{const}$ along a streamline or by noting that the flow speed gradients are driven by pressure gradients.

The flow speed outside the gap is negligible due to $t \ll R$, hence we can take the pressure at the bottom side to be uniformly p_0. The flow speed in the gap at a distance x from the axis of symmetry can be found from the conservation of mass applied on a concentric cylinder of radius x and height t, giving $2\pi x t \rho_w v(x) = \mu$. Hence,

$$v(x) = \frac{\mu}{2\pi x t \rho_w}.$$

Applying Bernoulli's principle, we get $p(x) + \rho_w v^2/2 = p_0$ and so

$$\Delta p = p_0 - p(x) = \frac{\rho_w v^2}{2} = \frac{1}{2\rho_w}\left(\frac{\mu}{2\pi x t}\right)^2.$$

We can hence find the force due to the pressure differences in the gap by integrating from $x = r$ to $x = R$. First note that the resulting force will be pointing vertically up, because $p(x) < p_0$. Integrating,

$$\begin{aligned} F_1 &= \int_{x=r}^{x=R} 2\pi x \,\mathrm{d}x\, \Delta p \\ &= \frac{\mu^2}{4\pi t^2 \rho_w} \int_{x=r}^{x=R} \frac{\mathrm{d}x}{x} \\ &= \frac{\mu^2}{4\pi t^2 \rho_w} \ln\left(\frac{R}{r}\right). \end{aligned}$$

Note that the water entering through the pipe will slow down, pushing the disk further down. The net force from this, however, turns out to be negligible due to the condition $r \gg t$. To see this, one can argue that the said force is of order $\mu v_{\text{pipe}} \sim \mu\mu/(\rho_w r^2) \sim \mu^2/(\rho_w r^2) \ll F_1$.

Further, we have the gravitational force $F_g = -mg = -\pi R^2 h \rho_s g$ pulling the disk down and buoyancy force $F_b = \pi R^2 h \rho_w g$ pushing disk up. The force balance then reads $F_g + F_b + F_1 = 0$. Solving the equation, we find

$$\mu = 2\pi R t \sqrt{\frac{h\rho_w(\rho_s - \rho_w)}{\ln(R/r)}}\, g.$$

Grading:

- Speed dependence $v(x)$ from continuity condition **0.5 pts**
- Dynamic pressure from Bernoulli equation **0.5 pts**
- Express force by integrating dynamic pressure over disk area. **0.5 pts**
- Explanation why the jet's impact force can be ignored (or equivalently, an expression for the impact force that is carried along in the solution) **0.5 pts**
- Disk's weight and buoyancy force **0.5 pts**
- Express answer from force balance **0.5 pts**

iv) *(0.5 points)* In the context of thermodynamics, entropy is defined in terms of its differential, such that the change in entropy of a system is given by $\mathrm{d}S = \mathrm{d}Q/T$, where $\mathrm{d}Q$ is the heat entering the system, and T its temperature. Further, entropy in reversible thermodynamic processes is a state function, i.e. it only depends on the current (equilibrium) thermodynamical state of the system. This means that when calculating the entropy difference of one mole of vapour and liquid, the temperature at which the phase transition took place does not affect the final result.

As such, it's most convenient to consider the two final states as only differing by the liquid undergoing condensation at $t_0 = 100\,^\circ\mathrm{C}$. The final temperature is $t_0 = 100\,^\circ\mathrm{C}$ because that's when water vapour pressure is equal to p_0 (i.e. boiling temperature at atmospheric pressure). This corresponds to a heat of $\Delta Q = mL = 1\,\mathrm{mol} \cdot ML$ entering the vapour system, compared to the liquid one. Hence, the entropy difference between one mole of vapour and liquid is given by

$$\Delta S = \frac{\Delta Q}{T_0} = \frac{1\,\mathrm{mol} \cdot ML}{T_0} = 110\,\mathrm{J/K}.$$

Grading:

- $\Delta S = \frac{\Delta Q}{T}$ **0.1 pts**
- $\Delta Q = LM$ **0.1 pts**
- Understanding that $T = T_0$ **0.2 pts**
- Correct numerical answer **0.1 pts**

v) *(3 points)* Because the expansion of water is reversible, entropy is conserved. This means that the change in entropy due to the expansion of the vapour is balanced by the entropy change due to condensation. As discussed before, because entropy is a function of state, it's most convenient to calculate the entropy change by imagining n moles of water (n will later cancel out) cooling and expanding from T_t, p_t to T_0, p_0

and condensing rn moles of water at the end. r is found by demanding that $\Delta S = 0$ in this process.

The entropy change of the vapour is found by applying the first law of thermodynamics over a small temperature and pressure increment $\mathrm{d}T$, $\mathrm{d}p$:

$$\mathrm{d}S_{\text{vapour}} = \frac{\mathrm{d}Q}{T} = \frac{\mathrm{d}U + \mathrm{d}W}{T},$$

where $\mathrm{d}U = nc_v\,\mathrm{d}T$ is the change in internal energy of the vapour and $\mathrm{d}W = p\,\mathrm{d}V$ is the work done by the vapour. Importantly, we neglect the volume of water compared to the vapour, as that allows using the ideal gas to simplify the work differential. Using the ideal gas law, we then have

$$p\,\mathrm{d}V = p\,\mathrm{d}\left(\frac{nRT}{p}\right) = nR\,\mathrm{d}T - nRT\,\frac{\mathrm{d}p}{p}.$$

Hence,

$$\mathrm{d}S_{\text{vapour}} = n(c_v + R)\frac{\mathrm{d}T}{T} - nR\,\frac{\mathrm{d}p}{p}$$

and we can integrate to get

$$\Delta S_{\text{vapour}} = nc_p \ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right),$$

where $c_p = c_v + R = R(i+2)/2 = R\gamma/(\gamma-1)$ is the heat capacity at constant pressure. The net change in entropy is then

$$\begin{aligned}\Delta S_{\text{tot}} = 0 &= \Delta S_{\text{vapour}} + \Delta S_{\text{condens}} \\ &= \Delta S_{\text{vapour}} - rn\frac{ML}{T_0} \\ &= nc_p \ln\left(\frac{T_0}{T_1}\right) - nR\ln\left(\frac{p_0}{p_1}\right) - rn\frac{ML}{T_0}.\end{aligned}$$

Thus,

$$r = \frac{RT_0}{ML}\left(\ln\left(\frac{p_1}{p_0}\right) - \frac{\gamma}{\gamma-1}\ln\left(\frac{T_1}{T_0}\right)\right) = 0.114.$$

To find the mass flow rate, we start by finding the flow speed v of the outgoing liquid/vapour. Since we know r, we can apply energy conservation (necessary for the reversibility to hold) on a flowing water packet. This is most conveniently done by demanding energy conservation on the system as a whole. Suppose that in some time interval n moles of water vapour with a volume of V_t are created at the boiler. Then at the outlet, in the steady state, due to conservation of particles, $(1-r)n$ moles of water vapour at a volume of V_0, alongside rn moles of liquid water, are removed. The outflowing water has an additional kinetic energy of $n\mu v^2/2$. The energy change of the whole system due to both steps must cancel each-other out due to energy conservation. We can write this as

$$0 = W_{\text{in}} - W_{\text{out}} + U_{\text{in}} - U_{\text{out}} - nMv^2/2 = 0,$$

where $W_{\text{in}} = p_t V_t = nRT_t$ and $W_{\text{out}} = p_0 V_0 = (1-r)nRT_0$ are the works done by the incoming and outgoing packet and similarly, $U_{\text{in}} = c_v nT_t$ and $U_{\text{out}} = c_v nT_0 - rnML + rnRT_0$ are the internal energies of the incoming and outgoing packet. Notice that for the outgoing internal energy, we have an extra term of $rp_0V_0 = rnRT_0$. This is because

when talking about the latent heat of vaporisation, it includes the work done in order to expand the vapour into the volume it's supposed to occupy. Therefore, we should subtract the said work from the latent heat of vaporisation, in order for it to capture the actual change in the internal energy of the water. Combining everything, we get

$$\begin{aligned} 0 &= nRT_t - (1-r)nRT_0 + c_v nT_t \\ &\quad - c_v nT_0 + rnML - rnRT_0 - nMv^2/2 \\ &= (c_v + R)n(T_t - T_0) + rnML - nMv^2/2, \end{aligned}$$

and hence,

$$v = \sqrt{2\left(\frac{c_p \Delta T}{M} + rL\right)} = 906\,\mathrm{m/s}.$$

The density of air at the outlet is found from ideal gas law $\rho = p_0 M/(RT_0)$. The mass flow rate of water vapour is then

$$\mu_{\text{vapour}} = A\rho v = \frac{Ap_0 Mv}{RT_0},$$

but we also have liquid water flowing out, such that the total flow rate is given by

$$\mu = \frac{1}{1-r}\mu_{\text{vapour}} = \frac{Ap_0 Mv}{(1-r)RT_0} = 59\,\mathrm{g/s}.$$

Grading:

- *Finding r* **1.5 pts**
  - Idea that entropy is conserved. **0.2 pts**
  - Idea to calculate entropy change of the whole gas from T_t, P_t to T_0, P_0. **0.2 pts**
  - Equation $\mathrm{d}S = \frac{\mathrm{d}U + \mathrm{d}W}{T}$, and $\mathrm{d}S = \frac{c\,\mathrm{d}T + p\,\mathrm{d}V}{T}$ **0.1 pts, 0.1 pts**
  - Using the ideal gas law to calculate $\mathrm{d}S$ using two of the variables T, V, P. **0.1 pts**
  - Integrating **0.2 pts** and expressing entropy difference in terms of P and T **0.2 pts**
  - Using or deriving $c_v = 3R$. **0.1 pts**
  - Correct entropy change due to the phase change **0.2 pts**
  - Express the correct result for r **0.1 pts**
- *Finding μ* **1.5 pts**
  - Idea to use energy conservation along the flow. **0.2 pts**
  - Correct expression for work in the energy conservation. **0.2 pts**
  - Correct internal energy change. **0.5 pts**
  - Correct energy change due to kinetic energy. **0.1 pts**
  - Express correct v **0.1 pts**
  - Equation for gas flow rate $\mu = v\rho_g A$. **0.1 pts**
  - The ideal gas law with density **0.1 pts**
  - Correction factor $\frac{1}{1-r}$ for the whole mass flow. **0.2 pts**

*Note: using simple Bernoulli's equation, which does not consider phase changes, gave **0 pts** since the problem is about deriving what happens in that case.*