3. Rotating space station (13 points) — *Solution by Kaarel Hänni, grading schemes by Adam Warnerbring, Kaur Aare Saar, Maksim Pokrovskiy and ....*

i) *(0.5 points)* Letting ω be the angular velocity, the acceleration experienced by people on the "ground" is $\omega^2 R = \left(\frac{2\pi}{\tau}\right)^2 R$. Setting this equal to g gives

$$\left(\frac{2\pi}{\tau}\right)^2 R = g \implies \tau = 2\pi\sqrt{\frac{R}{g}} \approx 63.437\,\text{s}.$$

Grading:

- Finding the acceleration at the "ground" **0.2 pts**
- Correct expression for the period **0.2 pts**
- Correct numerical answer **0.1 pts**

ii) *(2 points)* Let us consider the motion of the ball in the non-rotating (inertial) frame of the center of mass of the spaceship. As the travel time is $t = \tau/2$, the throwing point on the ground will rotate by exactly half a circle between the ball being thrown and the ball being caught. In our inertial frame, the trajectory of the ball will just be a straight line between these two diametrically opposed points; this has length $2R$. In this inertial frame, the ball thus travels with a constant velocity $v_{\text{inertial}} = \frac{2R}{\tau/2} = \frac{4R}{\tau}$ (in the radial direction). The initial velocity vector in the rotating frame is the difference of the velocity vector in the inertial frame and the velocity vector of the throwing point in the rotating frame compared to the inertial frame. So the initial velocity vector in the rotating frame has a radial component of $\frac{4R}{\tau}$ and a tangential component of $\frac{2\pi R}{\tau}$. The throwing speed s is the magnitude of this vector, which is

$$s = \sqrt{\left(\frac{4R}{\tau}\right)^2 + \left(\frac{2\pi R}{\tau}\right)^2} = \frac{2R}{\tau}\sqrt{4+\pi^2} \approx 117\,\text{m/s}.$$

Grading:

- Finding the trajectory in an inertial frame $\frac{8}{15}$ **pts**
- Finding the radial velocity $\frac{8}{15}$ **pts**
- Finding the tangential velocity $\frac{8}{15}$ **pts**
- Correct velocity addition $\frac{4}{15}$ **pts**
- Correct numerical answer $\frac{2}{15}$ **pts**

iii) *(2 points)* When the balloon comes to a stop, it is in equilibrium in the rotating frame. An object at radius R' of mass m_1 has a fictitious radial (downward) force of $m_1\omega^2 R'$ acting on it in this frame. This force on the mass m is $m\omega^2(R-H+l)$. The upward force on the balloon is the difference between this force for the balloon and the buoyant force in the rotating frame, which is $\frac{4}{3}\pi r^3(M-M')\frac{n}{V}\omega^2(R-H) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$. Putting all this together, we can write down the condition that the radial force is 0 in equilibrium

$$m\omega^2(R-H+l) = \frac{4}{3}\pi r^3(M-M')\frac{P}{R_G T}\omega^2(R-H)$$

$$\implies m = \frac{\frac{4}{3}\pi r^3(M-M')P(R-H)}{R_G T(R-H+l)} \approx 110.8\,\text{kg}.$$

Grading: If l is neglected, i.e. the mass and balloon are considered to be at equal heights, this problem is marked out of **1.0 pts** maximum. Idea: balance of fictitious radial force and buoyant force **0.8 pts**. Correct expression for force on balloon **0.5**

pts and mass **0.2 pts**. Force balance correct **0.2 pts**. Expression for m correct **0.2 pts**. Numerical answer correct **0.1 pts**.

iv) *(1.5 points)* In the rotating frame, letting r be the distance from the axis of the cylinder, there is an effective radial potential of

$$\varphi(r) = \int_0^r (-\omega^2 x)\,\mathrm{d}x = -\frac{\omega^2 r^2}{2},$$

where we have chosen the potential zero level to be at $r = 0$. In other words, in this frame, there is a fictitious radial force of $m_1\omega^2 r = -m_1\frac{\mathrm{d}\varphi}{\mathrm{d}r}$ acting on an object of mass m_1. The rope takes a shape that minimizes this potential energy. For this part and the next, we will just be figuring out properties of a rope that minimizes this potential energy (and other than that, we can forget about the rotation). Consider cutting off a tiny piece of rope of length ℓ from point C, then pulling the rope tight at C and gluing it back together (doing work ℓT_C), then cutting the rope open at A, letting it slip by length ℓ (doing work $-\ell T_A$), and finally moving the tiny piece from point C to point A (doing work $\ell\lambda(\varphi(R) - \varphi(R-h)) = -\frac{\ell\lambda\omega^2 h(2R-h)}{2}$), filling the gap of length ℓ. The state of the rope is now the same as initially, so the total work done should be 0:

$$\ell T_C - \ell T_A - \frac{\ell\lambda\omega^2 h(2R-h)}{2} = 0,$$

from where

$$T_A - T_C = -\frac{\lambda 2\pi^2 h(2R-h)}{\tau^2}.$$

See *200 More Puzzling Physics Problems*, problem 78 (and its hint and solution) for a longer explanation of this idea.

Grading: Idea: move piece of rope from C to A/B **0.2 pts**. Idea: work done by tension **0.1 pts** and correct expression **0.3 pts**. Idea: change in radial potential in rotating frame OR kinetic energy in lab frame **0.1 pts** and correct expression **0.3 pts**. Total work done is zero **0.2 pts**. Correct answer for tension difference **0.3 pts**.

v) *(1.5 points)* Let's use the equilibrium condition that torque around the center of the cylinder is 0 (in the rotating frame) for the left half of the rope. Note that the fictitious force is radial, so that contributes nothing to the force. So the only contributions are from the tension in the rope on the two sides, so

$$RT_A\cos\alpha = (R-h)T_C \implies \frac{T_A}{T_C} = \frac{R-h}{R\cos\alpha}.$$

Grading: Idea: rope in equilibrium **0.2 pts**. Using zero total torque as condition **0.2 pts**. Eliminating radial force by choice of point **0.4 pts**. Some correct torque expression containing both tensions **0.2 pts**. Correct ratio **0.5 pts**.

vi) *(1.5 points)* Let the x-axis be the diameter AB, with coordinates chosen to be in meters, and with the coordinates of A being $(-R, 0)$. (So the coordinates of B are $(R, 0)$ and the coordinates of C are $(0, -(R-h))$. The unique parabola $y = ax^2 + bx + c$ that goes through these three points has $b = 0$ from symmetry across the y axis, $c = -(R-h)$ from condsidering the point C, and then $a = -\frac{R-h}{R^2}$ from considering the point A. The derivative at A is $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-R} = 2a\cdot R = -\frac{2(R-h)}{R}$, from which

$$\cos\alpha = \frac{-\mathrm{d}y}{\sqrt{(\mathrm{d}y)^2 + (\mathrm{d}x)^2}} = \frac{1}{\sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}} = \frac{1}{\sqrt{1 + \frac{R^2}{4(R-h)^2}}} \approx 0.7106.$$

Using the equations from parts (iv) and (v), we now have a system of equations in two unknowns:

$$\begin{cases} T_A - T_C = -\dfrac{\lambda 2\pi^2 h(2R-h)}{\tau^2} \\ RT_A\cdot 0.7106 = (R-h)T_C \end{cases}.$$

It remains to solve this system of equations. The second equation gives $T_A = T_C \frac{R-h}{R \cdot 0.7106} \approx T_C \cdot 0.7107$. Plugging this into the first equation then gives

$$T_C = \frac{\lambda 2\pi^2 h(2R-h)}{\tau^2(1-0.7107)} \approx 12631\,\text{N}.$$

Grading: Some correct parabola given some choice of axes **0.3 pts**. Correct expression for $\cos\alpha$ **0.2 pts** and correct numerical value **0.2 pts**. System of equations using expressions found in iv) and v), even if they are incorrect, **0.2 pts**. Correct expression for tension **0.4 pts**. Solved only for T_A: full marks with penalty $-$**0.1 pts**. Correct numerical value for T_C **0.2 pts**.

vii) *(2 points)* The rotating charge on the walls is making the spaceship into a solenoid carrying current $I = \frac{Q}{\tau}$. Inside of the spaceship, this creates a constant axial magnetic field of magnitude

$$B = \mu_0 \frac{I}{L} = \mu_0 \frac{Q}{\tau L}.$$

The force this creates on the charged ball is $-qvB = -q\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L}$ in the radial upward direction. For the ball to hover above the "ground" motionlessly, the acceleration created by this radial force has to be equal to the centripetal acceleration $\omega^2 R = \frac{(2\pi)^2 R}{\tau^2}$:

$$-\frac{q}{m}\frac{2\pi R}{\tau}\mu_0\frac{Q}{\tau L} = \frac{(2\pi)^2 R}{\tau^2} \implies \frac{q}{m} = -\frac{2\pi L}{\mu_0 Q}.$$

Grading:

- **Method 1:**
  - Finding the current $\frac{4}{15}$ **pts**
  - Finding the magnetic field **0.4 pts**
  - Finding the upward force on the charged ball $\frac{2}{3}$ **pts**
  - Writing out the centripetal acceleration of the ball $\frac{4}{30}$ **pts**
  - Correct relation between force and acceleration $\frac{4}{15}$ **pts**
  - Correct final expression $\frac{4}{15}$ **pts**
  - Wrong sign of the charge $-\frac{4}{15}$ **pts**
- **Method 2:**
  - Finding the current $\frac{4}{15}$ **pts**
  - Finding the magnetic field **0.4 pts**
  - Finding the upward force on the charged ball $\frac{2}{3}$ **pts**
  - Writing out the force balance in the rotating frame (with the correct expression of the upward force) **0.4 pts**
  - Correct final expression $\frac{4}{15}$ **pts**
  - Wrong sign of the charge $-\frac{4}{15}$ **pts**

viii) *(2 points)* Consider a charge q_1 at rest at radius r in the rotating frame. The electromagnetic force applied to it is frame-independently $q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$. This charge is not moving in the rotating frame, so in its frame, the force applied to it by the magnetic field must be 0, and so the electric field measured in its frame must satisfy $q_1 E = q_1\frac{2\pi r}{\tau}\mu_0\frac{Q}{\tau L}$, from where

$$\vec{E} = \frac{2\pi\mu_0 Q}{\tau^2 L}\vec{r}.$$

We now have an expression for $\vec{E}$ at each point in the rotating frame; what remains is to evaluate $\oint \vec{E}\cdot \mathrm{d}\vec{A}$. For the sake of variety, we will demonstrate two ways to evaluate this integral.

For the first option, note that our $\vec{E} = \text{const} \cdot \vec{r}$ is exactly the electric field of a uniformly charged cylinder (with the correctly chosen charge density), and so we could equivalently find the same integral around such a uniformly charged cylinder, which Gauss' theorem gives as $V\rho/\varepsilon_0 = V \cdot c_1$. We can find an expression for the constant c_1 by considering the simple case where the surface we are dealing with is a cylinder of width 1 m itself, in which case the integral is

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = 1\,\mathrm{m} \cdot 2\pi r \frac{2\pi\mu_0 Q}{\tau^2 L} r = V \frac{4\pi\mu_0 Q}{\tau^2 L}.$$

Hence, $c_1 = \frac{4\pi\mu_0 Q}{\tau^2 L}$, and so

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = V \frac{4\pi\mu_0 Q}{\tau^2 L}.$$

To briefly describe a second option for evaluating this integral, note that by partitioning a 3D body into volume slices, with each slice bounded by an area element $\mathrm{d}\vec{A}$ and with cylindrical radius vector $\vec{r}$, and with the volume of each slice being $\vec{r} \cdot \mathrm{d}\vec{A}/2$, we get that the volume of the body is the sum of volumes of all such slices,

$$V = \oint \frac{1}{2} \vec{r} \cdot \mathrm{d}\vec{A}.$$

This lets us evaluate the main integral as

$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \oint \frac{2\pi\mu_0 Q}{\tau^2 L} \vec{r} \cdot \mathrm{d}\vec{A} = \frac{4\pi\mu_0 Q}{\tau^2 L} \oint \frac{1}{2} \vec{r} \cdot \mathrm{d}\vec{A} = \frac{4\pi\mu_0 Q}{\tau^2 L} V.$$

Grading:

- Arguing that in the rotating frame there is only an electric field **0.4 pts**
- Finding the electric field strength **0.4 pts**
- Choosing a suitable Gauss surface **0.4 pts**
- Evaluating the surface integral and obtaining the correct answer **0.8 pts**