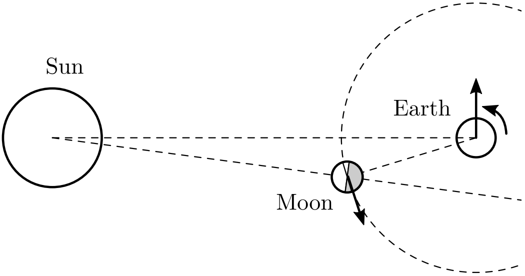
8. Planets (9 points) — *Solution by Taavet Kalda.*

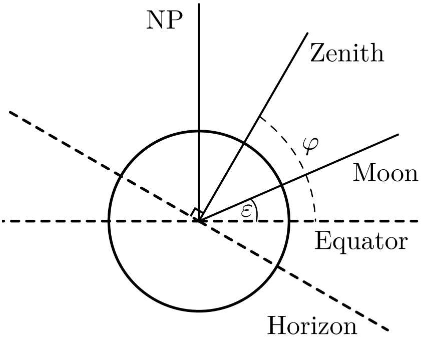
i) *(0.7 points)* A waxing crescent moon can be seen immediately after sunset, illustrated below



Grading:

- Figure with Moon between Earth and Sun **(0.3 pts)**
- All rotation directions consistent **(0.1 pts)**
- Correct answer A **(0.3 pts)**

ii) *(1.2 points)* Zenith in Tallinn forms an angle $\varphi = 59.5^\circ$ with the celestial equator. During a winter solstice, the Sun is $\varepsilon = 23.5^\circ$ below the celestial equator. Since full moon occurs when it's opposite to the Sun, Moon must be ε above the celestial equator. Further, the Moon is at its highest when it crosses the plane formed by the centre of Earth, the north pole (marked NP) and the ray pointing towards the zenith in Tallinn. A convenient way to imagine this is how during one day, stars and planets (alongside the Moon) stay roughly the same relative to eachother in the sky, but the sky as a whole rotates around the axis formed by the centre of Earth and north pole. This is depicted in the figure below, alongside with the celestial equator. From the figure, we can now read that the Moon is an angle $\varphi - \varepsilon$ from the zenith and hence the maximal culmination angle is $90° - \varphi + \varepsilon = 54.0°$.



Grading:

- Understanding of the positions of Tallinn, Earth and Moon at the maximum point **(0.5 pts)**
- Correct final expression **(0.6 pts)**
- Correct numerical value 54.0° **(0.1 pts)**
- Expression and value to Zenith 36.0° given instead **(0.5 pts)**

iii) *(1.2 points)* The flux of an object drops inverse squared to the distance from the said object.

Mars is closest/farthest to Earth when it aligns with the line formed by Earth and Sun. In both cases, Mars is at full phase. At its closest, Mars is $d_- = R_♂ - R_\oplus$ from Earth, and its farthest, $d_+ = R_♂ + R_\oplus$. In both cases, the same amount of sunlight reaches the Mars surface (because we assume Mars to always be $R_♂$ from the Sun). This, combined with phases being the same, gives us the ratio of illuminances to be

$$\frac{I_{\max}}{I_{\min}} = \frac{d_+^2}{d_-^2} = \left(\frac{R_♂ + R_\oplus}{R_♂ - R_\oplus}\right)^2 = 25.$$

Grading:

- Noting $I \propto 1/r^2$ **(0.2 pts)**
- Noting minimum/maximum correspond to $R_♂ \pm R_\oplus$ **(0.2 pts)**
- Noting Mars is at full phase in both positions **(0.2 pts)**
- Correct final expression **(0.5 pts)**
- Correct numerical value 25 **(0.1 pts)**

iv) *(1.2 points)* Looking from the Sun, the separation between Earth and Mars must change from 0 to π between the two described situations. The relative angular speed of the two planets is $\Delta\omega = 2\pi/T_{\oplus} - 2\pi/T_{\male}$. Mars' period can be found from Kepler's III law as $T_{\male} = T_{\oplus}\sqrt{(R_{\male}/R_{\oplus})^3}$. The time taken is thus

$$t = \frac{\pi}{\Delta\omega} = \frac{T_{\oplus}}{2}\frac{1}{1-(R_{\oplus}/R_{\male})^{3/2}} = 1.1\,\text{yrs}.$$

Grading:

- Expressing the relative angular speed $\Delta\omega$ **(0.3 pts)**
- Solving $T_{\male}$ in terms of $T_{\oplus}$ and orbital radii **(0.3 pts)**
- Correct final expression for half a period **(0.5 pts)**
- Correct numerical value 1.1 yrs **(0.1 pts)**

v) *(1.2 points)* The maximal angular separation between Sun and Venus, as seen from Earth, can be found as the angle between Earth-Sun ray and the ray that goes through Earth and is tangent to the orbit of Venus. From the right angled triangle, we get the seperation to be $\alpha = \arcsin(R_{\female}/R_{\oplus}) = 46.1^\circ$. We therefore see Venus for a duration of

$$1\,\text{day}\cdot\frac{\alpha}{2\pi} = 11\,050\,\text{s} = 3\text{hrs } 4\text{min}.$$

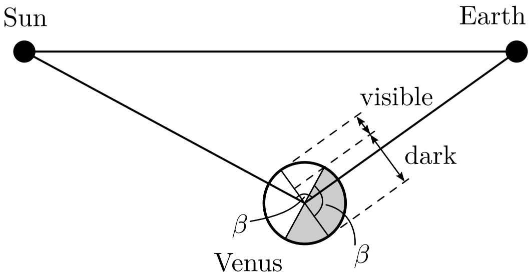
Grading:

- Angular separation is the angle between the Earth-Sun and Earth-Venus rays **(0.2 pts)**
- Expressing time as a ratio to a full rotation of the Earth **(0.2 pts)**
- Maximal when tangent to orbit **(0.4 pts)**
- Correct expression for α **(0.3 pts)**
- Correct numerical value for time $11\,050\,\text{s} =$ 3hrs 4min **(0.1 pts)**

vi) *(2.5 points)* As discussed in part iii), the luminosity of a planet is the product of its phase, φ (the fraction of its area which is illuminated), and the inverse squared distance to Earth. In other words,

$$\frac{I}{I_0} = \frac{\varphi}{L^2}.$$

In the figure below, the phase is simply the ratio of the lengths of the "visible" to the "dark" part.



From the geometry, we see that this is $(1+\cos\beta)/2$, where β is the angle between the Sun, Venus, and Earth. From cosine law, we get

$$R_{\oplus}^2 = R_{\venus}^2 + L^2 - 2R_{\venus}L\cos\beta$$

so

$$1+\cos\beta = 1 + \frac{R_{\venus}^2 - R_{\oplus}^2 + L^2}{2R_{\venus}L}.$$

Hence,

$$\begin{aligned}\frac{I}{I_0} &= \frac{R_{\venus}^2 - R_{\oplus}^2}{2R_{\venus}L^3} + \frac{1}{L^2} + \frac{1}{2R_{\venus}L} \\ &= -\frac{R_{\oplus}^2 - R_{\venus}^2}{2R_{\venus}}x^3 + x^2 + \frac{1}{2R_{\venus}}x,\end{aligned}$$

where we set $x = L^{-1}$.

Grading:

- Realizing I/I_0 is dependent on what fraction of Venus is illuminated from Earth's view **(0.4 pts)**
- Dependency of the form φ/L^2 **(0.4 pts)**
- Realizing illuminated area is proportional to $1+\cos\beta$ **(0.8 pts)**
- Expressing β using the cosine law **(0.2 pts)**
- Correct final expression **(0.7 pts)**

vii) *(1 point)* Our goal is to maximise $I(x)/I_0$ in the range $x \in [1/(R_{\oplus}+R_{\venus}), 1/(R_{\oplus}-R_{\venus})]$. $I(x)$ is a cubic that starts off from 0 at $x=0$ and extends all the way to negative infinity at $x \longrightarrow \infty$. We find the extrema by setting the derivative to zero

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{I}{I_0}\right) = 0 = -3\frac{R_{\oplus}^2 - R_{\venus}^2}{2R_{\venus}}x^2 + 2x + \frac{1}{2R_{\venus}}.$$

This is a quadratic whose solutions are

$$x_{\pm} = \frac{2R_{\venus} \pm \sqrt{3R_{\oplus}^2 + R_{\venus}^2}}{3(R_{\oplus}^2 - R_{\venus})^2}.$$

The negative solution is smaller than 0, and hence doesn't interest us. Now, the positive solution must correspond to a maximum due to the function tending to negative infinity at big values. After careful rearranging, the positive solution simplifies to

$$1/x_+ = -2R_♀ + \sqrt{3R_⊕^2 + R_♀^2} = 0.436\,\mathrm{AU},$$

which safely falls in the range of $[R_⊕ - R_♀, R_⊕ + R_♀]$ and hence is the distance of maximal intensity, $L_0 = 1/x_+ = 0.436\,\mathrm{AU}$.

We find the maximal angular distance between the Sun and Venus using cosine law as

$$\alpha_0 = \arccos\left(\frac{R_⊕^2 + L_0^2 - R_♀^2}{2L_0R_⊕}\right) = 39.6^\circ.$$

Grading:

- Noting maxima occurs at zero of derivative **(0.5 pts)**
- Correct final expression for L **(0.2 pts)**
- Correct numerical value for L **(0.1 pts)**
- Numerical answer within range $[0.28\,\mathrm{au}, 1.72\,\mathrm{au}]$ **(0.1 pts)**
- Correct numerical value for the angular distance 39.6° **(0.1 pts)**