1. Curling (8 points) — *Solution by Jaan Kalda, Oskar Vallhagen, grading schemes by Oskar Vallhagen.*

i) *(1 point)* As the stone is gliding over the ice everywhere on the contact surface, the friction force is $F_{fx} = \mu mg$, i.e. the acceleration is μg, directed in the negative x-direction (with the x-direction pointing towards the target stone). Thus, we get

$$\frac{dv_s}{dt} = -\mu g \Rightarrow v_s = v_0 - \mu g t.$$

Grading:

- Correct friction force **(0.5 pts)**
- Correct acceleration **(0.2 pts)**
- correct v_s **(0.3 pts)**

ii) *(1 point)* The simplest way to obtain the final velocity is to use energy conservation, noting that the friction force does a work $W_f = \mu mgs$. Thus, we get

$$\frac{mv_0^2}{2} = \frac{mv_{\text{hit}}^2}{2} + \mu mgs \Rightarrow v_{\text{hit}} = \sqrt{v_0^2 - 2\mu gs}.$$

Grading:

- Idea of using energy conservation **(0.2 pts)**
- Calculating the work done by the friction force **(0.3 pts)**
- Correct total energy conservation equation v_s **(0.3 pts)**
- Correct v_{hit} **(0.2 pts)**

Alternatively, one can first calculate the time t_{hit} before the stone hits the opponents stone. Integrating the expression for v_s up to this time gives

$$s = v_0 t_{\text{hit}} - \mu g \frac{t_{\text{hit}}^2}{2}.$$

Solving for t_{hit}, we get

$$t_{\text{hit}} = \frac{v_0}{\mu g} - \sqrt{\frac{v_0^2}{\mu^2 g^2} - \frac{2s}{\mu g}}$$

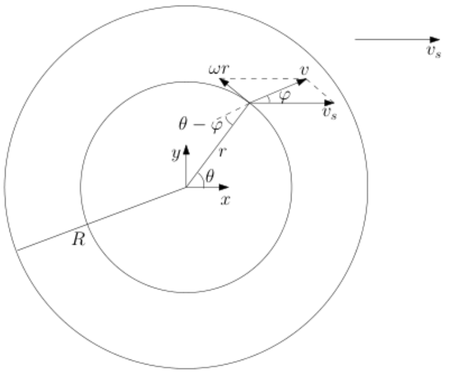
(the solution with the plus sign gives a negative velocity). Inserting back into the expression for v_s and simplifying gives

$$v_{\text{hit}} = \sqrt{v_0^2 - 2\mu gs}.$$

Grading:

- Integration of the expression for v_s **(0.4 pts)**
- Solving for t_{hit} **(0.4 pts)**
- Inserting into expression for v_s and final answer **(0.2 pts)**

iii) *(2 points)* Consider the motion of a point at the ring in contact with the ice at an angle θ from the x-axis, moving at a velocity v as depicted in the figure below. The components of this motion along the x and y axes can be seen to be $v_x = v_s - \omega r \sin\theta$ and $v_y = \omega r \cos\theta$, respectively.



The friction force per unit length has the magnitude

$$F_f' = \frac{\mu m g}{2\pi r}$$

everywhere on the circle in contact with the ice, as before, but is now directed opposite to the local velocity v rather than the sliding velocity v_s. The component of the force density in the x-direction thus becomes

$$F_{fx}' = -F_f' \cos\varphi = -F_f' \frac{v_x}{v}.$$

From this point on, there are two ways to work out the mathematical calculations. **First**, consider an infinitesimal arc of the contact ring of central angle $\mathrm{d}\theta$; the friction force exerted on it has magnitude $\mathrm{d}F_f = \frac{\mu m g}{2\pi}\mathrm{d}\theta$. The total force is obviously directed antiparallel to the velocity of the stone as y-directional components of the acting on arcs at θ and $\pi - \theta$ cancel pairwise out. The projection of the friction force acting on our arc is given by

$$\mathrm{d}F_x = \frac{\mu m g}{2\pi}\cos\varphi \mathrm{d}\theta$$

Using the small parameter $\delta \equiv \omega r / v_s \approx \omega r / v \ll 1$, we conclude that $\varphi \ll 1$, hence we can use approximation

$$\mathrm{d}F_x \approx \frac{\mu m g}{2\pi}\left(1 - \frac{1}{2}\varphi^2\right)\mathrm{d}\theta.$$

The angle φ can be found from the sine theorem assuming $\sin\varphi \approx \varphi$: $\varphi \approx \sin\left(\frac{\pi}{2} - \theta\right)\frac{\omega r}{v} \approx \cos\theta \frac{\omega r}{v_s} \equiv \delta \cos\theta$. Alternatively, a similarly accurate expression cab be found by noting that

$$\varphi \approx \tan\varphi = \frac{v_y}{v_x} \approx \frac{v_y}{v_s} = \frac{\omega r \cos\theta}{v_s}.$$

Thus,

$$\mathrm{d}F_x = \frac{\mu m g}{2\pi}\left(1 - \frac{\delta^2}{2}\cos^2\theta\right)\mathrm{d}\theta.$$

Recognising that $\int_0^{2\pi}\cos^2\theta d\theta = \pi$, this can be integrated to obtain

$$F_x = \mu m g\left(1 - \frac{\omega^2 r^2}{4 v_s^2}\right)$$

so that the final answer is

$$\Delta F_x = -\frac{\mu m g \omega^2 r^2}{4 v_s^2}.$$

The second way is by continuing

$$\begin{aligned}
F_{fx}' &= F_f' \frac{v_x}{v} \\
&= F_f' \frac{v_s - \omega r \sin\theta}{\sqrt{v_s^2 - 2 v_s \omega r \sin\theta + \omega^2 r^2}} \\
&= F_f' \frac{1 - \delta \sin\theta}{\sqrt{1 - 2\delta \sin\theta + \delta^2}}.
\end{aligned}$$

To second order in δ, we get, using the approximation suggested in the problem text for $x = -2\delta \sin\theta + \delta^2$ and doing some elementary algebra,

$$F'_{fx} \approx F'_f(1 - \frac{1}{2}\delta^2 \cos^2\theta).$$

The total friction force can now be obtained by integrating over the length element of size $rd\theta$ along the ring in contact with the ice, yielding, after simplifying and re-inserting the expression for F'_f,

$$F_{fx} = \int_0^{2\pi} F'_{fx} r d\theta = \mu mg \left(1 - \frac{1}{4}\delta^2\right).$$

Grading:

- Understanding that the friction force is everywhere antiparallel with the local velocity **(0.4 pts)**
- Calculating the magnitude of the force per unit length **(0.3 pts)**
- Correct expression for the x component **(0.2 pts)**

First solution:

- Approximate expression for F'_{fx} or $\mathrm{d}F_{fx}$ for small φ **(0.3 pts)**
- Expressing φ in terms of δ and θ **(0.5 pts)**
- Correct integral and final answer **(0.3 pts)**

Second solution:

- Expressing F'_{fx} or $\mathrm{d}F_{fx}$ in terms of δ and θ **(0.3 pts)**
- Approximate expression for F'_{fx} or $\mathrm{d}F_{fx}$ for small δ **(0.5 pts)**
- Correct integral and final answer **(0.3 pts)**

iv) *(2 points)*For the torque τ' per unit length (around the central axis of the stone), we see from the figure that the angle between the local force per unit length and the radius is $\theta - \varphi$, yielding

$$\tau' = F'_f r \sin(\theta - \varphi).$$

From here on, there are again two ways to continue. **First,** we can use Taylor series and keep only the linear term: $\sin(\theta - \varphi) \approx \sin\theta - \varphi\cos\theta$; by using the previously obtained expression $\varphi \approx \delta\cos\theta$, we can express

$$\mathrm{d}\tau = \mu mgr(\sin\theta - \delta\cos^2\theta)\frac{\mathrm{d}\theta}{2\pi}$$

integration of which yields

$$\tau = -\frac{\mu mg\omega r^2}{2v_s}.$$

The second way is to write

$$\begin{aligned}\tau' &= F'_f r \sin(\theta - \varphi)\\ &= F'_f r(\sin\theta\cos\varphi - \cos\theta\sin\varphi)\\ &= F'_f r(\sin\theta\frac{v_x}{v} - \cos\theta\frac{v_y}{v}),\end{aligned}$$

which, using the above expressions for v_x and v_y, can be simplified to

$$\tau' = F'_f r \frac{\sin\theta - \delta}{\sqrt{1 - 2\delta\sin\theta + \delta^2}}.$$

Applying the same approximation as in the previous part and keeping only the first, now leading, order terms in δ, we get

$$\tau' \approx F_f' r(\sin\theta - \delta\cos^2\theta).$$

Integrating to find the total torque, we get

$$\tau = \int_0^{2\pi} \tau' r d\theta = -\frac{1}{2}\delta\mu m g r.$$

Grading:

- Finding the angle between the local force per unit length and the radius **(0.5 pts)**
- Correct expression for τ'**(0.4 pts)**

First solution:

- Approximate expression for τ' or dτ for small φ **(0.3 pts)**
- Expressing φ in terms of δ and θ **(0.5 pts)**
- Correct integral and final answer **(0.3 pts)**

Second solution:

- Expressing τ' or dτ in terms of δ and θ **(0.3 pts)**
- Approximate expression for τ' or dτ for small δ **(0.5 pts)**
- Correct integral and final answer **(0.3 pts)**

v) *(2 points)* The equation describing the rotation is, now expressed in terms of ω and v_s rather than δ,

$$I\frac{d\omega}{dt} = \tau = -\frac{1}{2}\mu m g r^2 \frac{\omega}{v_s},$$

where the moment of inertia is $I \approx \frac{1}{2}mR^2$. As the correction to F_{fx} is only of second order in δ, to the first, leading order approximation, we may neglect the effect of rotation on F_{fx}. Thus, the sliding speed v_s has the same time dependenca as in part i). The equation above can then be separated as

$$\int_{\omega_0}^{\omega_{\text{hit}}} \frac{d\omega}{\omega} = \int_0^{t_{\text{hit}}} -\mu g \frac{r^2}{R^2} \frac{dt}{v_0 - \mu g t} \Rightarrow$$

$$\ln\frac{\omega_{\text{hit}}}{\omega_0} = \frac{r^2}{R^2} \ln\frac{v_0 - \mu g t_{\text{hit}}}{v_0},$$

which, recognising that $v_0 - \mu g t_{\text{hit}} = v_{\text{hit}}$, yields

$$\omega_{\text{hit}} = \omega_0 \left(\frac{v_{\text{hit}}}{v_0}\right)^{\frac{r^2}{R^2}}.$$

Grading:

- Correct equation of motion for the rotation **(0.5 pts)**
- Realising that v_s is unaffected by the rotation to leading order **(0.5 pts)**
- Using separation of variabels **(0.5 pts)**
- Final answer **(0.5 pts)**