2. Nitrogen explosion (8 points) — *Solution by Päivo Simson, grading schemes by Päivo Simson and ....*

i) *(1.5 points)* As shown in the first figure of the problem, the sphere floats so that exactly half of it is submerged in water. When studying buoyancy, the mass of nitrogen gas can be considered negligibly small compared to the masses of liquid nitrogen and the plastic sphere. According to Archimedes' principle, we have

$$\rho_n V_n + \rho_p V_p = \rho_w V_w,$$

where V_n, V_p, and V_w are the volumes of liquid nitrogen, plastic, and displaced water, respectively. We have

$$\frac{1}{2} \cdot \frac{4\pi r^3 \rho_n}{3} + d \cdot 4\pi r^2 \rho_p = \frac{1}{2} \cdot \frac{4\pi r^3 \rho_w}{3},$$

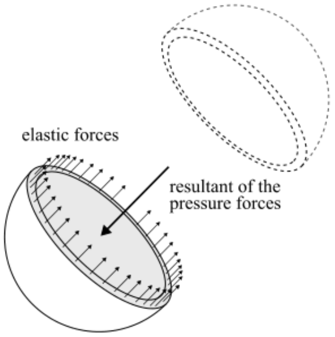
and solving for d we get

$$d = \frac{r}{6} \cdot \frac{\rho_w - \rho_n}{\rho_p} = 1.5\,\mathrm{mm}.$$

Grading:

- Archimedes' principle **(0.3 pts)**
- Correct volumes of liquid nitrogen, plastic, and displaced water **(0.6 pts)**
- Correctly solving for d **(0.3 pts)**
- Correct answer $d = 1.5\,\mathrm{mm}$ **(0.3 pts)**

ii) *(1.5 points)*The sphere will explode when the pressure difference $p_2 - p_a$ is such that the maximum tensile strength of the plastic is reached. Let's look at only one-half of the sphere and study the balance of the forces.



The resultant force due to pressure is $(p_2 - p_a)\pi r^2$. This must always be balanced by the elastic forces in the plastic. Right before the explosion, we have the equality

$$(p_2 - p_a)\pi r^2 = \sigma 2\pi r d,$$

where $2\pi r d$ is the cross-sectional area of the plastic. Now solving for p_2 we get

$$p_2 = p_a + \frac{2\sigma d}{r} = 1.1 \cdot 10^6\,\mathrm{Pa}.$$

Grading:

- Qualitative understanding of the conditions for the sphere to explode - resultant pressure force equals the total tensile strength **(0.4 pts)**
- The idea of breaking the sphere in half and analysing the forces acting on only one side of the sphere **(0.4 pts)**
- Correct condition for the sphere to explode: $(p_2 - p_a)\pi r^2 = \sigma 2\pi r d$ **(0.4 pts)**

- Correct final expression and answer **(0.3 pts)**

iii) *(1.5 points)* During boiling at constant pressure, the temperature of the liquid does not change, even though the liquid is gaining energy all the time. If however the pressure changes, as we have in our problem, then so does the boiling point, following exactly the phase transition line on the phase diagram. Since we know the final pressure p_2, we can simply read the corresponding temperature of the diagram:

$$p_2 = 1.1 \cdot 10^6\,\mathrm{Pa} \implies T_2 = 106\,\mathrm{K}.$$

Grading:

- Correctly stating that the process $1-2$ follows the phase transition line. **(1.0 pts)**
- Reading the correct temperature from the graph **(0.5 pts)**

iv) *(1.5 points)*If p, V, and T are known, the mass of the nitrogen gas can be calculated using the ideal gas equation:

$$pV = \frac{m}{M}RT \implies m = \frac{pVM}{RT}.$$

Since we know the initial and final states, the mass Δm of nitrogen that evaporated between these states is

$$\Delta m = m_2 - m_1 =$$

$$= \frac{2\pi r^3 M}{3R}\left(\frac{p_2}{T_2} - \frac{p_1}{T_1}\right) = 0.065\,\mathrm{kg}.$$

Grading:

- The idea of calculating the mass of nitrogen gas in the initial and final state **(0.3 pts)**
- Realizing that ideal gas law can be used to calculate the mass of nitrogen gas **(0.3 pts)**
- Correct ideal gas law **(0.3 pts)**
- Correctly using the molar mass **(0.3 pts)**
- Correct calculation and correct final answer **(0.3 pts)**

v) *(2 points)* Neglecting the heat capacity of the plastic and the heat flux through the upper half of the sphere, we only need to consider the heat of evaporation:

$$Q_e = \lambda\Delta m = 1{,}31 \cdot 10^4\,\mathrm{J},$$

and the heat for raising the temperature of the liquid nitrogen:

$$Q_l = c_v m_l (T_2 - T_1) =$$

$$= \frac{2}{3}\pi r^3 \rho_n c_v (T_2 - T_1) = 9{,}7 \cdot 10^4\,\mathrm{J}.$$

The heat for rising the temperature of the nitrogen gas can be neglected, as the mass of the gas is very small compared to the mass of liquid nitrogen ($\approx 1.7\,\mathrm{kg}$). The total heat $Q_e + Q_l$ is taken from the water through the plastic. The average temperature of nitrogen is $T_n = (T_1 + T_2)/2 = 91.7\,\mathrm{K}$, and the average temperature difference between water and nitrogen is $T_w - T_n = 201.5\,\mathrm{K}$. The heat flux through the plastic is

$$q = k \cdot \frac{T_w - T_n}{d}.$$

The total amount of heat carried through the lower half of the sphere during time Δt is

$$\Delta Q = qA\Delta t = Q_e + Q_l,$$

where $A = 2\pi r^2$ is half of the surface area of the sphere. Finally, solving for Δt, we have the estimated time it takes for the sphere to explode:

$$\Delta t = \frac{d \cdot (Q_e + Q_l)}{2\pi r^2 k\,(T_w - T_n)} = 15.5\,\mathrm{s}.$$

Grading:

- Understanding that $Q_e + Q_l$ must be equal to the total heat received from the water **(0.3 pts)**
- Correct heat of evaporation **(0.2 pts)**
- Correct heat for rising the temperature of liquid nitrogen **(0.2 pts)**
- Correctly using the average temperature of nitrogen **(0.2 pts)**
- Correctly using the average temperature difference **(0.2 pts)**
- Correct expression for the heat flux q though the plastic **(0.3 pts)**
- Correct expression for the total heat ΔQ received through the plastic **(0.3 pts)**
- Correct final expression for calculating the time till explosion **(0.3 pts)**
  Give full points for this part if the final expression and the answer are correct.