3. WOBBLE (8 points) — *Solution by Taavet Kalda, grading schemes by ….*

i) *(2.5 points)* The farther away a planet is from the host star, the bigger its orbital period. In a fixed period, we then expect planet A to make more rotations around the star than planet B. The wobble effect comes from both the star and the planets orbiting around their common barycentre (centre of mass). The effect of one planet makes the star undergo a circular motion with some radius x and frequency ω. The effect of two planets is additive, so the overall motion is the sum of two circular motions with radius' x_A and x_B with different angular frequencies ω_A and ω_B.

In the figure, we find the centre of circular motion of the star from the centre of the envelope (by for example using a ruler to find the diameter and then the centre-point). From there, we measure that lower frequency component covers an angle of $\alpha = 290^\circ$ within the measurement period $t = 10\,\mathrm{yr}$. The higher frequency component, in the meantime, undergoes 7.5 full rotations with respect to the lower frequency component. Hence, $\omega_B t = \alpha$ and $(\omega_A - \omega_B)t/(2\pi) = 7.5$. This yields

$$
\begin{aligned}
T_B &= \frac{2\pi}{\omega_B} = \frac{2\pi t}{\alpha} = 12.4\,\mathrm{yr}, \\
T_A &= \frac{2\pi}{\omega_A} = \frac{1}{\frac{1}{T_B} + \frac{7.5}{t}} = 1.20\,\mathrm{yr}.
\end{aligned}
$$

Grading:

- Realising that the higher frequency component is due to planet A and the lower frequency component is due to planet B **(0.5 pts)**
- The higher frequency component undergoes 7.5 full rotations **(0.4 pts)**
- The lower frequency component covers an angle of 290°, or alternatively, $\frac{7.5}{9} \cdot 2\pi$ **(0.4 pts)**
- Finding T_A **(0.8 pts)**
- Finding T_B **(0.4 pts)**

ii) *(2.5 points)* From the figure, we measure the extrema of the distance of the star from the origin as $l_1 = 1.6 \times 10^5\,\mathrm{km}$ and $l_2 = 2.8 \times 10^5\,\mathrm{km}$. The extrema corresponds to when the two circular motions are pointing in the same direction or the opposite direction. In other words, the distances are $l_1 = x_B - x_A$ and $l_2 = x_B + x_A$. Rearranging, $x_A = (l_2 - l_1)/2 = 6 \times 10^4\,\mathrm{km}$ and $x_B = 2.2 \times 10^5\,\mathrm{km}$. To infer the masses, we need to study the dynamics of the system.

First, we can assume that the semi-major axis of the planets are much bigger than x. Indeed, $a_A/x_A = 4000 \gg 1$. Either from force balance between gravity and centrifugal force, or through Kepler's Third Law, we have $4\pi^2/(GM) = T_A^2/a_A^3$ so the star's mass is

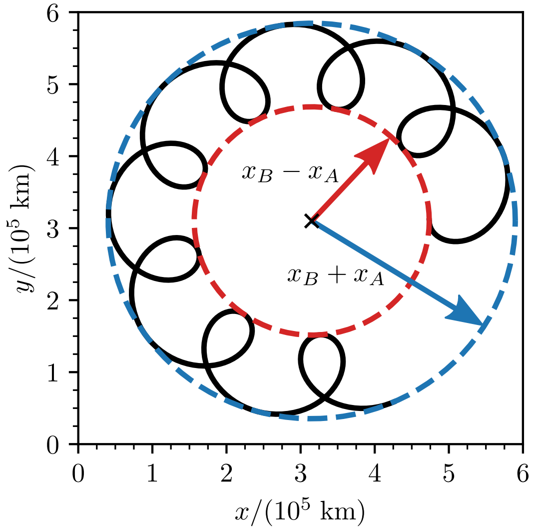
$$
M = \frac{4\pi^2 a_A^3}{GT_A^2} = 4.4 \times 10^{30}\,\mathrm{kg}.
$$

The planet's mass can be inferred from the property that planet and star rotate around their common centre of mass: $Mx_A = m_A a_A$. Hence,

$$m_A = M\frac{x_A}{a_A} = 1.2 \times 10^{27}\,\text{kg}.$$

Grading:

- Expressing $4\pi^2/(GM) = T_A^2/a_A^3$ **(0.6 pts)**
- Finding M **(0.4 pts)**
- Realising that the position of the centre of mass of A and M is conserved (with respect to B) **(0.5 pts)**
- Inferring equation $Mx_A = m_A a_A$ **(0.5 pts)**
- Finding x_A from the trajectory **(0.3 pts)**
- Finding m_A **(0.2 pts)**



iii) *(1 point)* Once again, from Kepler's Third law, we get the semi-major axis of planet B to be

$$a_B = \sqrt[3]{\frac{T_B^2 GM}{4\pi^2}} = 1.0 \times 10^9\,\text{km}.$$

Similarly to before, we get the mass as

$$m_B = M\frac{x_B}{a_B} = 9.3 \times 10^{26}\,\text{kg}.$$

Both of the planets are gas giants, similar to Jupiter (0.63 and 0.50 Jupiter masses respectively). This makes sense, as this method is more sensitive to higher mass exoplanets with bigger orbits.

Grading:

- Finding a_B **(0.5 pts)**
- Finding m_B **(0.5 pts)**

iv) *(2 points)* The equations describing the system still hold, but reading off the frequencies is trickier. Nevertheless, we can still locate the origin and measure the extreme distances $l_1 = 1.20 \times 10^5\,\mathrm{km}$ and $l_2 = 1.85 \times 10^5\,\mathrm{km}$. We also note that the higher frequency component (corresponding to planet A) has a bigger radius than the lower frequency one (planet B). Hence, $l_1 = x_A - x_B$ and $l_2 = x_A + x_B$, and so $x_A = (l_1 + l_2)/2 = 1.5 \times 10^5\,\mathrm{km}$, $x_B = (l_2 - l_1)/2 = 3.3 \times 10^4\,\mathrm{km}$.

The higher frequency component does 3 full turns and, measuring from the figure, an extra 280° on top of it. This gives $T_A = t/(3 + 280°/360°) = 2.65\,\mathrm{yr}$. The lower frequency component, meanwhile, starts and ends from the farthest away point, and oscillates through the closest point 3 times. Hence it goes through 3 full rotations. Thus,

$$\left(\frac{2\pi}{T_A} - \frac{2\pi}{T_B}\right) t = 6\pi,$$

and so

$$T_B = \frac{1}{\frac{1}{T_A} - \frac{3}{t}} = 12.9\,\mathrm{yr}.$$

By using the equations from the previous parts, we get

$$M = \frac{4\pi^2 a_A^3}{G T_A^2} = 6.8 \times 10^{29}\,\mathrm{kg},$$
$$m_A = M\frac{x_A}{a_A} = 5.1 \times 10^{26}\,\mathrm{kg},$$
$$m_B = M x_B \sqrt[3]{\frac{4\pi^2}{T_B^2 G M}} = 3.9 \times 10^{25}\,\mathrm{kg}.$$

The first planet is once again a gas giant (0.26 Jupiter masses) while the second one is a "Super-Earth" (6.4 Earth masses).

Grading:

- Finding T_A **(0.4 pts)**
- Finding T_B **(0.4 pts)**
- Finding M **(0.3 pts)**
- Finding m_A **(0.4 pts)**
- Finding m_B **(0.5 pts)**