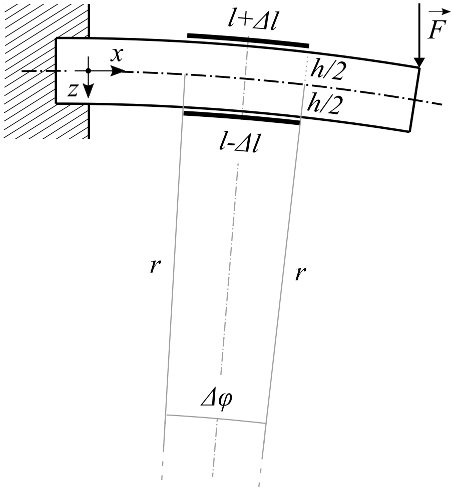
5. Force sensor (5 points) — *Solution by Päivo Simson, grading schemes by ....*

i) *(2 points)* To find the elongation Δl, we need to know the curvature radius r of the beam at $x = L/2$. The torque created by the force F at an arbitrary point x is $F \cdot (L - x)$. This must be balanced by the bending moment $M(x)$ of the beam. In the middle of the beam we have

$$M = \frac{EI}{r} = F\frac{L}{2} \implies r = \frac{2EI}{FL}.$$

It is easy to see that if the upper wires elongate by Δl, the lower ones shorten by the same amount. Now we need to relate the curvature radius r with the elongation Δl.



The arc length of a circle is $\Delta s = r\Delta\phi$. Knowing this, we have from the above figure

$$l + \Delta l = \left(r + \frac{h}{2}\right)\Delta\phi,$$

$$l - \Delta l = \left(r - \frac{h}{2}\right)\Delta\phi.$$

By dividing the above equations, we get an equation for Δl that is easily solved:

$$\frac{l + \Delta l}{l - \Delta l} = \frac{2r + h}{2r - h} \implies \Delta l = \frac{lh}{2r}.$$

Combining this with the expression for r we have

$$\Delta l = \frac{FLlh}{4EI}.$$

Grading:

- Correct moment equation $EI/r = FL/2$ at the center of the beam **(0.6 pts)**
- Correct geometric relations for $l \pm \Delta l$ based on $\Delta s = r\Delta\phi$ **(0.6 pts)**
- correctly solving for Δl **(0.6 pts)**
- correctly expressing the final answer **(0.2 pts)**

ii) *(1 point)* Let ρ and S be the resistivity and the cross-sectional area of the wires, respectively. The initial resistance R_0 of all the wires is

$$R_0 = \frac{\rho l}{S} = \frac{\rho l^2}{lS},$$

where lS is the volume of the wire that remains constant during the deformation. Assuming $\Delta l << l$ we have

$$R_1 = \frac{\rho(l+\Delta l)^2}{lS} \approx \frac{\rho l(1+2\Delta l/l)}{S} = R_0 + \Delta R,$$

$$R_2 = \frac{\rho(l-\Delta l)^2}{lS} \approx \frac{\rho l(1-2\Delta l/l)}{S} = R_0 - \Delta R,$$

where

$$\Delta R = \frac{2\rho\Delta l}{S} = 2R_0\frac{\Delta l}{l} = R_0\frac{FLh}{2EI}.$$

Grading:

- $R = \rho l/S$ **(0.3 pts)**
- Correct approximations **(0.3 pts)**
- Correctly calculating the resistances and the change in resistance using the above relations **(0.4 pts)**

iii) *(2 points)* The total resistance R of the circuit is

$$R = \frac{(R_1+R_2)(R_1+R_2)}{2R_1+2R_2} = \frac{R_1+R_2}{2}$$

Since both sides of the circuit have the same total resistance $R_1 + R_2$, the current is also the same. The voltmeter reading is therefore

$$V = \frac{I}{2}R_2 - \frac{I}{2}R_1 = \frac{U}{2R}(R_1 - R_2) =$$

$$= U\frac{R_1 - R_2}{R_1 + R_2} = U\frac{\Delta R}{R_0} = U\frac{FLh}{2EI}.$$

From this, we finally have

$$F = \frac{2EIV}{LhU}.$$

Grading:

- Correct total resistance of the Wheatstone circuit **(0.5 pts)**
- Assumes that the resistance of the Voltmeter is infinite **(0.5 pts)**
- Correctly calculating the potential difference V as a function of the change in the resistance **(0.5 pts)**
- Expressing the final answer in the form given in the solution **(0.5 pts)**