6. STRING-COUPLED MASSES (5 points) — *Solution by Aleksi Kononen and Jaan Kalda, grading schemes by Aleksi Kononen.*

i) *(2 points)* For in-phase oscillations, the string connecting the masses always remains parallel to the x-axis. In that case, the motion is confined to the yz-plane and is identical to a pendulum with length $r\sin\theta$ hung from the axis of rotation:

$$\omega_1 = \sqrt{\frac{g}{r\sin\theta}}$$

Grading:

- Stating $\sum F_x = 0$, or stating that the connecting string remains parallel to the x-axis **(0.6 pts)**
- Realising the effective pendulum length is $r\sin\theta$, i.e. the axis of rotation is the line connecting the wall points **(0.8 pts)**
- Correct expression for ω_1 **(0.6 pts)**

ii) *(3 points)* Now suppose one of the balls moves along the y axis to a distance ym and the other ball — by $-y$. The length of the horizontal string remains constant, hence $y^2 + (r-x)^2 = r^2$, where x denotes the displacement of the ball in x-direction. For small displacements, we can neglect the smallest term x^2, hence $x = y^2/2r$. The length of the slanted string remains also constant, hence $(r\sin\theta - z)^2 + (r\cos\theta + x)^2 + y^2 = r^2$, where z denotes the vertical displacement. Neglecting here the smallest terms z^2 and x^2, we obtain $2rz\sin\theta = 2rx\cos\theta + y^2 = y^2(1+\cos\theta)$, hence

$$z = \frac{y^2}{2r}\frac{1+\cos\theta}{\sin\theta}.$$

Now we can write down energy conservation law: $\frac{1}{2}\dot{y}^2 + gz = const$, where dot denotes the time derivative. Differentiating this over time we obtain

$$\ddot{y} = -\frac{gy}{r}\frac{1+\cos\theta}{\sin\theta}$$

so that

$$\omega_2 = \sqrt{|\ddot{y}/y|} = \omega_1\sqrt{1+\cos\theta}.$$

Grading:

- Equation for a cylinder surface **(0.4 pts)**
- Equation for a sphere surface **(0.4 pts)**
- Both surface equations correct and coordinates consistent **(0.2 pts)**
- Realising x^2- and z^2-terms are negligible, and not discarding y^2 **(0.2 pts)**
- Solving z in terms of y **(0.5 pts)**
- Energy conservation between z and $\dot{y}$ **(0.3 pts)**
- Solving relation between $\ddot{y}$ and y **(0.6 pts)**
- Correct expression for ω_2 **(0.3 pts)**
- ω_2 correctly in terms of ω_1 **(0.1 pts)**

Alternative solution: Let one of the masses (mass A) be displaced by y and the other by $-y$. Let T' be the tension in the two outermost strings (equal magnitudes by symmetry) and T in the connecting string. Newton's law along the y-axis for mass A then yields

$$-T'\frac{y}{r} - T\frac{y}{r} = m\ddot{y},$$

i.e.

$$\ddot{y} = -\frac{T'+T}{mr}y.$$

Without oscillations, $T_0' = mg/\sin\theta$ and $T_0 = T_0'\cos\theta$. Let us write $T' = T_0' + \Delta T'$ and $T = T_0 + \Delta T$, i.e.

$$T' + T = mg\frac{1+\cos\theta}{\sin\theta} + \Delta T' + \Delta T.$$

Let us make some estimates. Clearly, $|\Delta T| \sim |\Delta T'|$. The maximum of $\Delta T'$ occurs when the mass swings by its lowest point with velocity $\dot{y}$, where its trajectory has a radius of curvature $R \sim r\sin\theta$. Moreover, small normal mode oscillations imply $y = A\cos\omega_2 t$ for some amplitude $A \ll r\sin\theta$, i.e. $|\dot{y}| \leq A\omega_2$. Finally, assume $\omega_2 = k\omega_1$ holds for some $k \sim 1$. We have

$$|\Delta T'| \sim \frac{m\dot{y}^2}{R} \sim \frac{mk^2A^2g}{r^2\sin^2\theta} = k^2mg\left(\frac{A}{r\sin\theta}\right)^2,$$

i.e. quadratically smaller in $(A/r\sin\theta)$ than T_0'. Thus, $T'+T \approx mg\frac{1+\cos\theta}{\sin\theta}$ and

$$\omega_2 = \sqrt{\frac{T'+T}{mr}} = \omega_1\sqrt{1+\cos\theta}.$$

(The result shows $k \leq \sqrt{2}$, i.e. the assumption holds.) **Grading:**

- Force equation, either along y or full vector form **(0.4 pts)**
- Solving relation between $\ddot{y}$ and y in terms of tension forces **(0.4 pts)**
- Unperturbed values for T' and T **(0.4 pts)**
- Reasonable estimate of $|\Delta T'|$ **(0.6 pts)**
- Showing $|\Delta T'|$ is quadratically smaller than T' **(0.8 pts)**
- Correct expression for ω_2 **(0.3 pts)**
- ω_2 correctly in terms of ω_1 **(0.1 pts)**
- **Remark:** Accept solution with assumption $\Delta T'/T' \ll 1$ without estimates