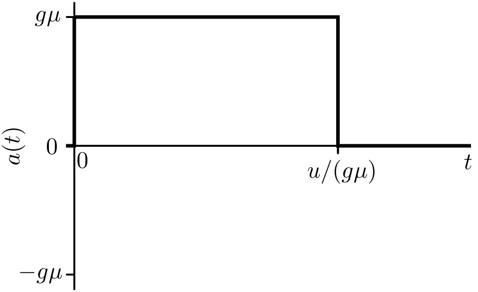
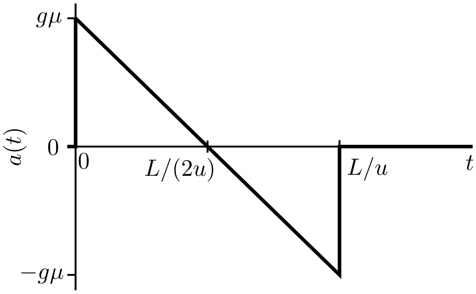
7. A PACK OF PAPERS (8 points) — *Solution by Taavet Kalda, grading schemes by Martin Rahe, Isak Fleig and Joonas Kalda.*

i) *(2 points)* Clearly, when u is very small, the pack gets dragged along with the bottommost sheet, so the speed v of the pack starts from 0, and ends up with u. Further, there is almost no slippage between the pack and the bottom sheet. The acceleration of the top sheet is then $g\mu$ as long as its being accelerated, and 0 otherwise. The acceleration lasts for a duration of $u/(g\mu)$.



Conversely at very large v, the bottommost sheet gets pulled out so fast that the pack ends with very little speed. As the sheets gets pulled from under, the acceleration of the pack drops from $u/(g\mu)$ to $-u/(g\mu)$ over a duration of L/u.



Grading:

- Correct initial accelerations **(0.2 pts)**
- Correct final accelerations **(0.2 pts)**
- Correct durations of acceleration **(0.4 pts)**
- Correct graph shapes **(0.6 pts)**
- Physical explanations for the graph shapes **(0.6 pts)**
  (half of the points in each subsection are given for part (a) and half for part (b))

ii) *(3 points)* The problem is most conveniently analysed in a frame moving with the bottom-most sheet with speed u. In the new frame, $v' = v - u$ and $x' = x - ut$. Let the centre-point of the bottom-most sheet be $x' = 0$ and let's track the movement of the pack using its right-most edge which will start at $x' = L/2$ with a speed of $v' = -u$ and lose contact with the bottom sheet at $x' = -L/2$.

When the pack is in contact with the bottom sheet and in the process of slipping, the pack has a normal force of $N_1 = mg(L/2 + x')/L$ with the bottom-most sheet, and $N_2 = mg(L/2 - x')/L$ with the table. Friction force will then result in a net force of $F = N_1\mu - N_2\mu = 2mg\mu x'/L$. The acceleration of the pack is thus $a' = 2g\mu x'/L$. Note that this equation is that of a spring with a negative spring coefficient $k = 2g\mu/L$ (which we will still denote with a positive k value, only that the force repels from the equilibrium). Nevertheless, conservation of energy still holds, $v'^2/2 - kx'^2/2 = \text{const}$. The most critical point to overcome is at $x' = 0$ (after that the force starts helping the pack), which makes sense because then most of the bottom-most paper is out from underneath the pack, and the friction force from the table is dominant. The condition for this is simply $v'^2 - kx'^2 > 0$ or in other words, $u^2 - kL^2/4 > 0$. This gives

$$u_{\min} = \frac{\sqrt{k}L}{2} = \sqrt{\frac{g\mu L}{2}}.$$

Grading:

- switching to the reference frame of the bottom-most sheet **(0.4 pts)**
- Correct normal forces with bottom-most sheet and table **(0.6 pts)**
- Correct net force on pack **(0.2 pts)**
- Identifying the spring coefficient **(0.3 pts)**
- Conservation of energy equation **(0.5 pts)**
- Identifying critical point $x = 0$ **(0.6 pts)**
- Deriving u_{min} **(0.4 pts)**

iii) *(1 point)* As before, conservation of energy holds. This means that when the bottom sheet is separated at $x' = -L/2$, the potential energy is the same as when $x' = L/2$ so the speed of the pack is still $-u$. Surprisingly, the final speed of the pack in the table frame is therefore 0.

Grading:

- Potential energy when the sheet is separated is the same as when Sandra starts pulling **(0.6 pts)**
- Correct final speed in the bottom-most sheet reference frame **(0.2 pts)**
- Correct final speed in the table reference frame **(0.2 pts)**

iv) *(2 points)* The goal is to find the overall distance by which the pack slides in the table frame. Equivalently, we can find the duration t_1 for which it slides, because then it will have shifted by $-L$ in the paper frame, and the paper frame has shifted by ut_1 with respect to the table frame. The overall shift is then $l = ut_1 - L$.

The equation of motion for the pack was $a' = kx'$ with $k = 2g\mu/L$. The solution for this is, instead of sine and cosine, hyperbolic sine and cosine. Seeing from the initial conditions, a possible solution could be $x' = x_0 \sinh(\omega(t - \Delta t))$. This indeed gives $v' = \omega x_0 \sinh(\omega(t - \Delta t))$ and $a' = \omega^2 x'$ such that $\omega = \sqrt{k}$ satisfies the equation of motion. We find the values for x_0 and Δt from the initial conditions at $t = 0$ so that $x' = -x_0 \sinh(\omega\Delta t) = L/2$ and $v' = x_0\omega\cosh(\omega\Delta t) = -u$. From here, we get

$$\Delta t = \frac{1}{\omega}\tanh^{-1}\left(\frac{L\omega}{2u}\right),$$

$$x_0 = -\frac{L_0}{2\sinh(\omega\Delta t)}.$$

Now, from the form of x', it's clear that it will pass $x' = 0$ at $t = \Delta t$, and reach $x' = -L/2$ at $t = 2\Delta t$. Therefore, $t_1 = 2\Delta t$ and the minimal distance between the paper pack and the edge has to be

$$l = ut_1 - L = \frac{2u}{\omega}\tanh^{-1}\left(\frac{L\omega}{2u}\right) - L$$
$$= L\left(\frac{2u}{L\omega}\tanh^{-1}\left(\frac{L\omega}{2u}\right) - 1\right).$$

Grading:

- Stating the formula $l = ut_1 - L$ **(0.2 pts)**
- Stating the equation of motion $a' = kx'$ **(0.2 pts)**
- Solution for the equation of motion **(0.4 pts)**
- Correct values for x_0 and Δt **(0.6 pts)**
- Justification for $t_1 = 2\Delta t$ **(0.3 pts)**
- Correct result for the minimal distance l **(0.3 pts)**-