8. Connected charges (8 points) — *Solution by Jaan Kalda, grading schemes by Jānis Cīmurs.*

i) *(2 points)* As the ball A enters the electric field, the force qE is exerted on it, hence the acceleration of the system of two balls is $a_1 = qE/2m$; the acceleration remains constant until the ball A exits the field. This will happen when $L = a_1 t_1^2/2$, hence $t_1 = \sqrt{2L/a} = 2\sqrt{Lm/qE}$ At that moment, the acceleration of the ball A becomes zero, but at the very same moment, ball B enters the field. The force qE exerted on it will give it twice as big an acceleration as before, $a_2 = qE/m$, and the string becomes loose. So, the speed of the ball B is now bigger than that of the ball A, and so they will eventually collide. The time t_2 spent by the ball B in the field can be expressed as $t_2 = (v_f - v_i)/a_2$, where $v_i = \sqrt{qEL/m}$ is its initial speed, and $v_f = \sqrt{3qEL/m}$ — the final speed. So, $t_2 = \sqrt{Lm/qE}(\sqrt{3} - 1)$. At the moment when the ball B exits the field, the distance between the balls is $s = L - (v_f - v_i)t_2/2 = v_i t_2 = L(\sqrt{3} - 1)$, and the difference between the velocities of the balls is $\Delta v = v_f - v_i = \sqrt{qEL/m}(\sqrt{3}-1)$, hence the time remaining until the collision $t_3 = s/\Delta v = \sqrt{Lm/qE}$. Thus, the moment of collision $t = t_1 + t_2 + t_3 = \sqrt{Lm/qE}(\sqrt{3} + 2)$.

Grading:

- Initially both charges move with the same acceleration **(0.2 pts)**
- When A escapes the field region A moves with constant speed, B moves with higher acceleration **(0.2 pts)**
- When B escapes, both move with constant speed **(0.1 pts)**
- Correct a_1 **(0.2 pts)**
- Correct t_1 **(0.2 pts)**
- Correct a_2 **(0.1 pts)**
- Correct v_i **(0.2 pts)**
- Correct v_f **(0.2 pts)**
- Correct t_2 **(0.2 pts)**
- Correct s **(0.2 pts)**
- Correct t_3 and t **(0.2 pts)**

ii) *(2 points)* **First solution:** The centre of mass moves with constant acceleration $a_1 = qE/2m$, and has travelled distance $s_{CM} = L - l/2 + l/(2\sqrt{2}) = 0.8200L = a_1 T^2/2$, hence $T = 1.811\sqrt{\frac{mL}{qE}}$.

Grading:

- Use $s_{CM} = a_1 T^2/2$ for centre of mass **(0.5 pts)**
- Correct a_1 **(0.5 pts)**
- Correct s_{CM} **(0.6 pts)**
- Correct T **(0.4 pts)**
- If you assume small angle oscillation to calculate time necessary for the string to align with the x-axis. **(up to 1.4 pts)**

iii) *(2 points)* **First solution:** Due to the conservation of y-directional momentum, the y-components of the balls are always equal. At $t = T$, the string is parallel to the x-axis; since it is not stretchable, x-components of the balls are equal, too, at that moment. Hence, the speeds are equal, and energy conservation law states $EqL = mv^2$ so that $v = \sqrt{\frac{qEL}{m}}$.

Grading:

- Using energy conservation law **(1.0 pts)**
- Kinetic energy of both masses **(0.6 pts)**
- Work done by electric field **(0.2 pts)**
- Correct v **(0.2 pts)**

Alternative solution: The time period $0 < t < T$ is most conveniently analysed in the centre-of-mass frame. In that frame, there is a force of inertia equal to $F_i = -ma_1 = -qE/2$. Hence, both balls perform non-linear oscillations around the centre of mass (it is nonlinear because the amplitude is not small). The speed of the ball at $t = T$ can be found from the energy conservation law. At $t = 0$, the speed is zero so that the full energy is due to the potential energy in the homogeneous force field $qE/2$: $\Delta U = \frac{qE}{2}(\frac{l}{2} - \frac{l}{2\sqrt{2}}) = \frac{qEL}{8}(2-\sqrt{2}) = \frac{mv_y^2}{2}$, hence $v_y^2 = \frac{qEl}{4m}(2-\sqrt{2}) = 0.1800\frac{qEL}{m}$. This gives us the velocity component in the y-direction; once we return to the laboratory frame, we need to add the x-component of the frame's velocity $v_x = a_1 T = \frac{qE}{2m}1.811\sqrt{\frac{mL}{qE}} = 0.9055\sqrt{\frac{qEL}{m}}$ so that $v = \sqrt{v_x^2 + v_y^2} = \sqrt{\frac{qEL}{m}}$.

Grading:

- Horizontal speed v_x **(0.4 pts)**
- Use energy balance or something similar to find rotational speed v_y **(0.4 pts)**
- Correct distance travelled **(0.6 pts)**
- Rotational speed v_y **(0.4 pts)**
- Pythagorean theorem **(0.2 pts)**

iv) *(2 points)* For the string to be parallel to the x-axis, the system must perform $n + \frac{1}{2}$ half-periods, where n is an integer. Since now the amplitude is small, the oscillations are harmonic, and the circular frequency $\omega = \sqrt{\frac{qE}{ml}}$ (we can use the formula for a pendulum of length $l/2$ and equivalent gravity field $g = qE/2m$). So, the travel time must be $\tau = \pi\sqrt{\frac{ml}{qE}}(n + \frac{1}{2})$. On the other hand, we can use the result from task ii: $\tau = \sqrt{\frac{ml}{qE}}\sqrt{4\lambda - 2 + 2\cos\phi}$, where $\lambda = L/l < 1$ and $\cos\phi \approx 1$. So, $\pi(n + \frac{1}{2}) = \sqrt{4\lambda}$. Since $\lambda < 1$, this equality can only be satisfied with $n = 0$, hence $\lambda = \frac{\pi^2}{16} = 0.6169$ so that $l = L/\lambda = 1.6211L$.

Grading:

- Correct circular frequency **(0.6 pts)**
- If circular frequency deviates from the right by factor $\sqrt{2}$ **(0.2 pts)**
- Correct rotation time with n **(0.6 pts)**
- Correct rotation time without n **(0.2 pts)**
- Correct travel time from ii **(0.4 pts)**
- Derivation of l **(0.4 pts)**