1. FOUR CHARGES (7 points) — *Solution by Päivo Simson, grading schemes by Päivo Simson, Oleg Košik and Lasse Franti.*

i) *(2 points)* Assuming $v << c$ and ignoring gravity, the initial total energy of the system is the sum of classical kinetic and potential energies

$$E_{total} = E_{kin} + E_{electric} =$$

$$E_{kin} + \frac{1}{2}\Sigma_{i\neq j}\, k\frac{q_i q_j}{r_{ij}} =$$

$$4\frac{mv_0^2}{2} + 4\frac{kq^2}{L} + 2\frac{kq^2}{\sqrt{2}L} =$$

$$= 4\left(\frac{mv_0^2}{2} + \frac{kq^2}{L}\frac{4+\sqrt{2}}{4}\right).$$

After the particles have moved infinitely far from each other, the potential energy becomes zero and the total energy is only kinetic:

$$E_{total} = 4\frac{mv_f^2}{2}.$$

Since the total energy does not change we have from the equality of these two expressions

$$v_f = \sqrt{v_0^2 + \frac{kq^2}{Lm}\frac{4+\sqrt{2}}{2}}.$$

Grading:

- Idea of using energy conservation **(0.2 pts)**
- Idea of symmetry and equality of quantities **(0.2 pts)**
- Idea of total energy as a sum of kinetic and electrostatic **(0.2 pts)**
- Formula for electrostatic energy including the two different distances **(0.3 pts)**
- Factor $\frac{1}{2}$ from pairings **(0.4 pts)**
- Final energy is purely kinetic **(0.4 pts)**
- Correct final answer **(0.3 pts)**
- If only dimensionless factors missing and final answer is reasonable **(0.2 pts)**

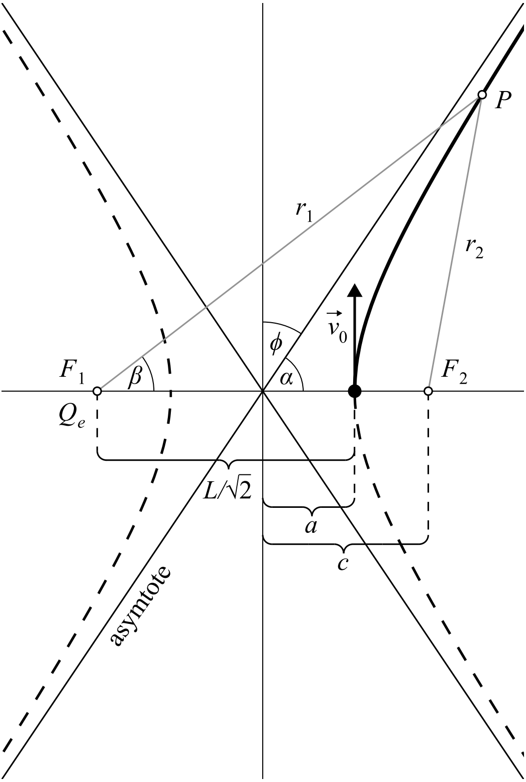
ii) *(5 points)*

Let the required angle be ϕ.

The expression for the total energy reveals that the particles behave effectively independent of each other as if they are only subject to a central field generated by a single effective charge

$$Q_e = q\frac{2\sqrt{2}+1}{4} \approx 0.96q,$$

fixed at the center of mass of the system, and initially at a distance $L/\sqrt{2}$ from each particle. This observation allows us to reduce the initial four-body problem into four independent (and identical) two-body problems involving only the spatially fixed charge Q_e and one moving charge q.The trajectories are therefore hyperbolas with one focus located at the position of the effective charge. Since the force is repulsive, the effective charge is at the focus F_1 as shown in the following figure.



Let's now calculate the asymptote angle α, since the required angle ϕ is just $\pi/2 - \alpha$.

A hyperbola is defined as a set of points

P, such that the absolute difference of the distances from P to two fixed points F_1 and F_2 (the foci) is constant. Let the distances be r_1 and r_2. From the figure it is easy to see that the constant is equal to $2a$: if we take the point P on the x-axis, then $r_1 - r_2 = c + a - (c - a) = 2a$.

To determine the asymptote angle α, let us look at the triangle F_1PF_2. From the law of cosines, we have

$$r_2^2 = r_1^2 + 4c^2 - 4cr_1 \cos\beta.$$

Solving for $\cos\beta$ and using $r_2 = r_1 - 2a$ we get

$$\cos\beta = \frac{r_1^2 - r_2^2 + 4c^2}{4cr_1} = \frac{a}{c} + \frac{c^2 - a^2}{cr_1}.$$

As $r_1 \to \infty$, then $\beta \to \alpha$, and we get the expression for the cosine of the asymptote angle:

$$\cos\alpha = \frac{a}{c}.$$

Now using $c = L/\sqrt{2} - a$ and knowing that

$$E = \frac{kqQ_e}{2a} = \frac{mv_0^2}{2} + \frac{\sqrt{2}kqQ_e}{L}$$

(the vis-viva equation for the hyperbola), from which

$$\frac{1}{a} = \frac{mv_0^2}{kqQ_e} + \frac{2\sqrt{2}}{L},$$

we finally get

$$\sin\phi = \cos\alpha = \frac{a}{c} = \frac{1}{\frac{L}{\sqrt{2}a} - 1} =$$

$$= \frac{1}{1 + \frac{Lmv_0^2}{\sqrt{2}kqQ_e}} = \frac{1}{1 + \frac{4Lmv_0^2}{kq^2(4+\sqrt{2})}}.$$

Grading:

- Effective repulsive central force $F = \frac{A}{r^2}$ pointing from the COM (explicit expression or statement is required) **(1 pts)**
- Realising hyperbolic motion **(0.5 pts)**
- Central charge is located at the correct focus **(0.5 pts)**

- Expression for the angle ϕ or α in terms of geometrical parameters e.g. $\cos\alpha = a/c$, $\tan\alpha = b/a$ **(1 pts)**
- Vis-viva equation OR angular momentum conservation **(1 pts)**
- Deriving final answer **(1 pts)**

Subpoints for angular momentum conservation:

- Mentioned that angular momentum is conserved **(0.3 pts)**
- Writes initial angular momentum around COM using L and v_0 **(0.3 pts)**
- Writes final angular momentum around COM using v_f **(0.4 pts)**

Variation 1.

If the student doesn't know the vis-viva equation for the hyperbola

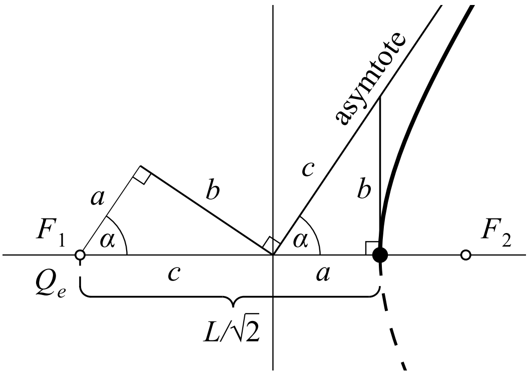
$$E = \frac{kqQ_e}{2a},$$

which differs from the corresponding equation for the ellipse only by the sign of the total energy, it can be derived as follows.

From the asymptote angle formula

$$\cos\alpha = \frac{a}{c}$$

the geometric relationships shown in the following figure follow immediately.



We see that the two right triangles on the figure are identical.

From the conservation of angular momentum (with respect to the point F_1) we have

$$m(c+a)v_0 = mbv_f \implies v_f = v_0\frac{c+a}{b},$$

and from the conservation of energy using $b^2 = c^2 - a^2$

$$\frac{kqQ_e}{a+c} = \frac{mv_f^2}{2} - \frac{mv_0^2}{2} =$$

$$= \frac{mv_0^2}{2}\left(\frac{(c+a)^2}{c^2-a^2} - 1\right) = \frac{mv_0^2}{2}\frac{2a}{c-a},$$

from which

$$\frac{mv_0^2}{2} = \frac{c-a}{2a} \cdot \frac{kqQ_e}{c+a}.$$

Inserting this into the total energy expression we finally get

$$E = kqQ_e\left(\frac{1}{2a} \cdot \frac{c-a}{c+a} + \frac{1}{c+a}\right) = \frac{kqQ_e}{2a}.$$

Variation 2.

Using the algebraic equation for the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and knowing that $b^2 = c^2 - a^2$, we get

$$\tan\alpha = \lim_{x\to\infty}\frac{y}{x} = \lim_{x\to\infty} b\sqrt{\frac{1}{a^2} - \frac{1}{x^2}} = \frac{b}{a}.$$

Variation 3.

The vis-viva equation step can be done using conservation of angular momentum around COM, as all the forces are radial.

$$\mathcal{L} = mv_0\frac{L}{\sqrt{2}} = mv_f b$$

since we already know $a + c = \frac{L}{\sqrt{2}}$, $c^2 = a^2 + b^2$, we can solve for the hyperbolas parameters.