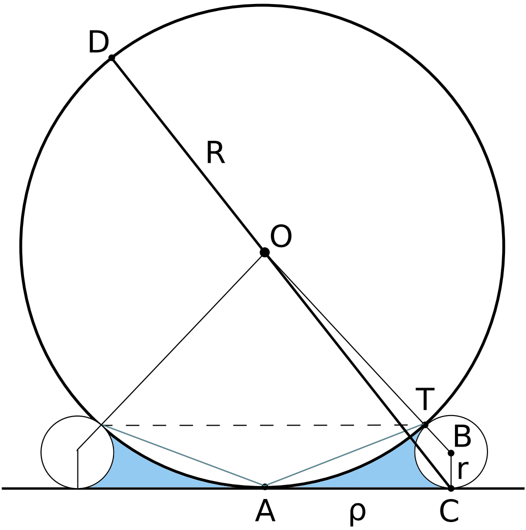
3. STICKY BALL (4 points) — *Solution by Jaan Kalda, Tudor Plopeanu, grading schemes by Eppu Leinonen.*



Let the radius of the neck be $\rho \ll R$ thus the force from surface tension at the contact of the sphere is negligible (angle to the horizontal ≈ 0). Then, the curvature radius of the meniscus $r \ll \rho$ can be found from the intersecting secants theorem applied on a point C on the bottom edge of the neck relative to the ball and approximating that the point T is (near) the intersection of CD and the circle: $2r \cdot 2R \approx \rho^2$. On the drawing, $AC^2 \approx CT \cdot CD$ and we approximate CT as $2r$, $R + r$ as R, and AC as ρ. The pressure difference between the water and the surrounding air is $\Delta p = \sigma / r$. Its total vertical component is equal to its amplitude times the vertical cross-section $S = \pi \rho^2$. As such, the total vertical force on the ball difference between the two cases (with and without water) is $\Delta F \approx \sigma \pi \rho^2 / r = 4\pi\sigma R$.

Grading: (preliminary) Note that differing sign conventions are tolerated

- Stating that the meniscus is (roughly) inverse spherical or usage of constant r to characterise the meniscus as inverse spherical **(0.5 pts)**
- $r \ll \rho \ll R$ **(0.1 pts)**
- Stating that surface tension from from contact is negligible or ΔF comes from the pressure difference (either explicitly or implicitly) **(0.5 pts)**
- $2r \cdot 2R \approx \rho^2$ **(0.7 pts)** (if not found, partial points can be earned as below)

- $CT \approx 2r$ **(0.1 pts)**
  - $R + r \approx R$ **(0.1 pts)**
  - $AC \approx \rho$ **(0.2 pts)**
  - $AC^2 \approx CT \cdot CD$ or statement of inter-esecting secants theorems **(0.3 pts)**
- $\Delta p = \sigma / r$ **(1 pts)** (if not found, partial points can be earned as below)
  - $\Delta p = \sigma(1/r_1 + 1/r_2)$ or any attempt to use a form of the Laplace-Young equation **(0.2 pts)**
  - $\Delta p = \sigma(1/\rho - 1/r)$ **(0.3 pts)**
- $S = \pi\rho^2$ (correct effective are for pressure) **(0.5 pts)**
- $\Delta F = S\Delta p$ **(0.5 pts)**
- $\Delta F \approx 4\pi\sigma R$ **(0.2 pts)** (0.1 pts for each of the following)
  - $\Delta F > 0$ or noting that the contact force increases
  - Correct dimensionless factor of 4π and correct dimensions (only if approach is correct)

Solution 2 by Eppu Leinonen

The position of the meniscus can be parameterised using the angle $\theta = \angle TOA$. The meniscus is small as the amount of water is small. Thus $|\theta| \ll 1$ and the radii can be approximated as $\rho = R\sin\theta \approx R\theta$ and $2r \approx R(1 - \cos\theta) \approx R\theta^2/2$. The change in normal force due to surface tension is $\Delta F_\sigma = 2\pi\sigma\rho\sin\theta \approx 2\pi\sigma R\theta^2 \approx 0$. Using the Young-Laplace equation $\Delta p = \sigma(1/\rho - 1/r) \approx \sigma(1/\theta - 4/\theta^2)$. Thus the total change in normal force due to pressure difference is $\Delta F_p = -S\Delta P = -\pi\rho^2\Delta P \approx \pi\sigma R(4-\theta) \approx 4\pi\sigma R^2$. I.e. $\Delta F = \Delta F_p + \Delta F_\sigma \approx \Delta F_p \approx 4\pi\sigma R$.

Grading: (preliminary) Note that differing sign conventions are tolerated

- Stating that the meniscus is (roughly) inverse spherical or usage of constant r to characterise the meniscus as inverse spherical **(0.5 pts)**
- $|\theta| \ll 1$ **(0.1 pts)**
- $\rho \approx R\theta$ **(0.2 pts)**
- $AC \approx \rho$ **(0.2 pts)**
- $r \approx R\theta^2/4$ **(0.3 pts)**
- $\Delta F_\sigma \propto \theta^2 \ll \Delta F_p$ or any statement that the force from surface tension is negligible or $\Delta F = \Delta F_p$ **(0.5 pts)**

- $\Delta p = \sigma/r$ **(1 pts)** (if not found, partial points can be earned as below)
  - $\Delta p = \sigma(1/r_1 + 1/r_2)$ or any attempt to use a form of the Laplace-Young equation **(0.2 pts)**
  - $\Delta p = \sigma(1/\rho - 1/r)$ **(0.3 pts)**
- $S = \pi\rho^2$ (correct effective area for pressure) **(0.5 pts)**
- $\Delta F_p = -S\Delta p$ **(0.5 pts)**
- $\Delta F \approx 4\pi\sigma R$ **(0.2 pts)** (0.1 pts for each of the following)
  - $\Delta F > 0$ or noting that the contact force increases
  - Correct dimensionless factor of 4π and correct dimensions (only if approach is correct)

Note: A common approach was to use the method of virtual displacement to solve for the force caused by the meniscus. However, it turns out that $\delta A_{lg} \propto \sqrt{r}\delta r$ (the area of the meniscus-air interface is roughly half of a spherical toroid with radii ρ and r) which means that the change in potential energy due to the change in surface area is negligible ($r \ll R$). Thus the virtual work done by lifting the sphere only counteracts the virtual work done by the pressure, which reduces the problem back to either solution 1 or solution 2. Thus no points are granted for just mentioning this approach and telling that $\delta U = \sigma\delta A_{lg}$.

Solution 3 using virtual displacement (by Jaan Kalda). Let us denote the contact area of water and plate with $A = \pi\rho^2$; then, the contact area of water and ball is also approximately A. Let the ball touch initially the plate, and then be raised by $\mathrm{d}x$. Since the volume of water is conserved, no work is made by atmospheric pressure, and $2r\mathrm{d}A = A\mathrm{d}x$, where $\mathrm{d}A$ is the change of the contact area; hence, $\frac{\mathrm{d}A}{\mathrm{d}x} = A/2r$. Meanwhile, for zero contact angle, the difference of surface energies at the air-solid interface, and at the water-solid interface equals to the surface energy of the water-air interface. This fact can be expressed in terms of the three surface tension coefficients denoted with σ_1, σ_2, and σ, respectively: $\sigma_1 - \sigma_2 = \sigma\cos\alpha = \sigma$. During our virtual displacement, the air-water interface remains almost constant; meanwhile, air-solid interface is increased by $2\mathrm{d}A$ (contributed equally by the ball and plate surfaces). Therefore, the surface energy is increased by $\mathrm{d}U = (\sigma_1 - \sigma_2)2\mathrm{d}A = 2\sigma\mathrm{d}A$. Now we can find force as $F = \frac{\mathrm{d}U}{\mathrm{d}x} = 2\sigma\frac{\mathrm{d}A}{\mathrm{d}x} = \sigma A/r$. This is the same expression we obtained from Young-Laplace equation. From this point on, the solution follows the steps made above.

Grading: (preliminary) Note that differing sign conventions are tolerated.

- Only the contact area between the solid and the liquid changes notably **(0.5 pts)**
- $\frac{\mathrm{d}A}{\mathrm{d}x} = A/2r$ **(0.5 pts)**
- $\sigma_1 - \sigma_2 = \sigma$ **(0.2 pts)**
- $\mathrm{d}U = 2(\sigma_1 - \sigma_2)\mathrm{d}A$ **(0.3 pts)**
- $F = \frac{\mathrm{d}U}{\mathrm{d}x}$ (only if A has been identified correctly) **(0.3 pts)**
- $F = \sigma A/r$ **(0.2 pts)**

These replace noting that $\Delta F_\sigma \approx 0$, $\Delta p = \sigma/r$ and $\Delta F = S\Delta p$ from the other solutions.