4. TOTALITY (8 points) — *Solution by Tudor Plopeanu, Taavet Kalda.* **Grading:** (preliminary)

- In in this task, numerical errors in the answers give **(-0.1 pts)** (as long as the answer is reasonable)

i) *(1.5 points)* As the distance from the Moon to the Sun is much larger than the distance from the Moon to the Earth, we can consider the speed of the Moon's shadow on Earth to be equal to the Moon's speed altogether. The speed of the Moon is the ratio of the circumference of its orbit to its period:

$$v_m = \frac{2\pi R_m}{T_m} = 1.02\,\mathrm{km/s}.$$

Because during the peak of the eclipse, the centre-points of Earth, the Moon and the Sun lie on the same line, the shadow of the Moon must move along the diameter of the Earth when viewed face-on. As such, the shadow travels a distance of $2r_e$ while it's still visible on Earth and the eclipse will be observable for time

$$T_{\text{ecl}} = \frac{2r_e}{v_m} = \frac{r_e}{\pi R_m} T_m$$
$$\approx 12.5 \times 10^3\,\text{s} = 3.46\,\text{h}.$$

Grading: (preliminary)

- Explains that the moons shadow can be approximated by the moons position **(0.5 pts)**
- Correct speed of the moon **(0.5 pts)**
- Correct final expression **(0.5 pts)**
- Minor mistake in final expression **(-0.2 pts)**

ii) *(1 point)* In absence of the Earth's rotation, the Moon's shadow would cover close to π radians. Over T_{ecl}, the Earth rotates by $2\pi \frac{T_{ecl}}{T_0}$ radians. The Earth's rotation goes "in the same direction" as the Moon's shadow, and therefore the total longitudinal reach of the eclipse is $|180(1 - 2T_{\text{ecl}}/T_0)| \approx 128$ degrees.

Grading: (preliminary)

- Figures out that the angle is π without the rotation of the earth **(0.3 pts)**
- Explains that the angle becomes smaller than π due to earths rotation **(0.3 pts)**
- Correct final formula and answer **(0.4 pts)**
- Thinks earths rotation goes against the moons rotation and obtains an angle $> \pi$ **(-0.3 pts)**

iii) *(1.5 points)* Right near the end of the eclipse, the Earth's rotation moves the surface under the Moon's shadow almost perpendicularly to the Moon's shadow's velocity. As such, the width of the Moon's shadow is equal to the distance travelled by the Moon's shadow for the duration of the totality at that point,

$$w_\lambda = v_m t_0 = 2\pi R_m \frac{t_0}{T_m} \approx 123\,\text{km}.$$

At the equator, the surface of the Earth is closer by r_e to the Moon compared to the point where the eclipse ends, serving to further increase the area of the full shadow. As the Moon covers an angle of approximately $\alpha = 2r_m/R_m$ radians in the sky, when getting closer to the moon by r_e, the width of its shadow will increase by roughly $\alpha r_e \approx$

57.7 km. As such, the total width of the shadow near the equator is around $w_{\mathrm{eq}} = v_m t_0 + \alpha r_e = 180\,\mathrm{km}$.

Grading: (preliminary)

- Realizes that the velocities are perpendicular at the end of the eclipse **(0.4 pts)**
- Calculates the width of the eclipse at some point on earth **(0.5 pts)**
- Correctly finds how to translate the width to the width at the equator **(0.5 pts)**
- Correct final answer **(0.1 pts)**

iv) *(1.5 points)* The peak of the eclipse occurs on the equator, firstly because that's where the Moon's shadow is the biggest due to it being closer to the Moon, but also because the Earth's rotation is the fastest there, and the rotation of the Earth serves to lengthen the effect of the eclipse. At the equator the surface of the Earth moves with speed $v_e = 2\pi r_e/T_0 = 0.46\,\mathrm{km/s}$. The angle between the velocity of the Earth's surface at the equator and the Moon's shadow's velocity is λ, so the relative speed of the Moon's shadow can be obtained from cosine theorem as

$$v_{\mathrm{rel}} = \sqrt{v_m^2 + v_e^2 - 2v_m v_e \cos\lambda} = 0.654\,\mathrm{km/s}.$$

The duration while the eclipse can be observed during the peak is thus $t_{\mathrm{eq}} = w_{\mathrm{eq}}/v_{\mathrm{rel}} = 276\,\mathrm{s} = 4.6\,\mathrm{min}$.

Grading: (preliminary)

- Shows that the eclipse must be observable the longest time at the equator **(0.5 pts)**
- Finds the relative velocity between the moon's shadow and the velocity of the surface of the earth **(0.5 pts)**
- From this finds the formula for the maximum duration of the eclipse **(0.5 pts)**
- Forgetting to take the angle between the velocities into account **(-0.3 pts)**

v) *(1 point)* Because $a \ll R_e$, we can treat the relative velocity between the Moon's shadow and Earth to be the same as in the previous subpart. The full shadow of the Moon on the Earth at the equator is a circle, and we can find the distance the Moon's shadow has to travel over the course of the totality at a point displaced by a using the Pythagorean Theorem as $2(w_{\mathrm{eq}}^2/4 - a^2)^{1/2}$. The eclipse is then observable for

$$\frac{1}{v_{\mathrm{rel}}}\sqrt{w_{\mathrm{eq}}^2 - 4a^2} = 230\,\mathrm{s} = 3.8\,\mathrm{min}.$$

Grading: (preliminary)

- Argues that the relative velocity is approximately the same as in the previous task **(0.5 pts)**
- Correct geometry and answer **(0.5 pts)**

vi) *(1.5 points)* With our assumptions, the typical width of an eclipse is $(w_\lambda + w_{\mathrm{eq}})/4$, and the typical length on the surface of the Earth is πr_e. As such, a typical eclipse covers an area equal to the product of the two above values. From there, we deduce the probability of an eclipse being noticeable from a given arbitrary point on Earth as the area spanned by the eclipse, $\pi r_e(w_\lambda + w_{eq})/4$, divided by the surface area of Earth, $4\pi r_e^2$. Then, the expected number of eclipses until one covers a given arbitrary point is the multiplicative inverse of the above probability. As eclipses occur, on average, every 18 months, the expected time for this to happen is

$$\frac{16 r_e}{w_\lambda + w_{\mathrm{eq}}} \cdot 18\,\mathrm{months} \approx 500\,\mathrm{years}.$$

Grading: (preliminary)

- Shows that the probability per eclipse is equal to the area covered by the eclipse divided by the total area of the earth **(0.5 pts)**
- Finds the area covered by one eclipse **(0.5 pts)**
- Correctly multiples the inverse probability with the duration between eclipses and finds the correct answer **(0.5 pts)**