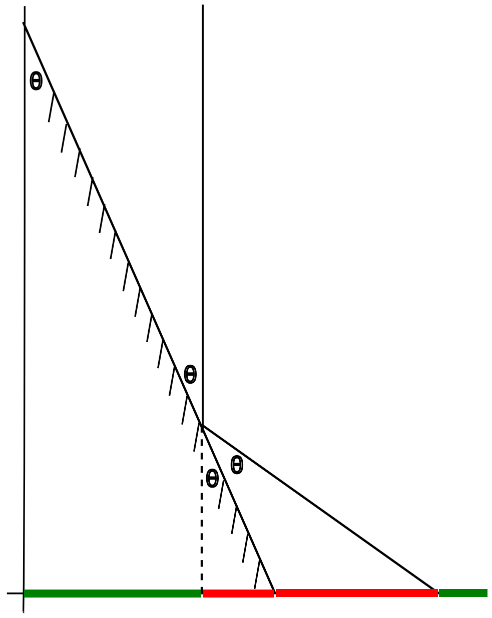
6. Cones (8 points) — *Solution by Tudor Plopeanu, grading schemes by Eppu Leinonen and Aleksi Kononen.*

i) *(2 points)* We will consider any extreme point on the red heart and its reflection. This may be easiest and most precise for the points furthest from the center of the picture, but any edge point of the red heart would suffice. Let θ be the cone's half apex angle. Let r be the measured ratio of the (angular) distance from the perceived image to the mirror and the perceived image to the original point. Then, $\frac{\tan\theta}{\tan 2\theta} = r$, so $\theta = \arccos\frac{1}{\sqrt{2(1-r)}}$, or alternatively $\arctan\sqrt{1-2r}$. Of course, the apex angle is 2θ, which is double the above.

For measured $r = 0.25$, we have the answer $2\theta \approx 1.22$ radians, or 71 degrees.



Grading: (preliminary)

- Stating that the image is formed by vertical rays as the camera is far away **(0.5 pts)**
- Correct explicit expression of θ with respect to r (or some other measurable variable) **(1 pts)** (if not found, partial points can be earned as below)
  - Correct geometrical figure **(0.2 pts)**
  - $r = \tan\theta/\tan 2\theta$ or equivalent or a correct implicit equation for θ **(0.5 pts)**
- Correct numerical answer $2\theta \in [65°, 76°]$ **(0.5 pts)** (only if approach is correct)
  - If only θ is given and not 2θ **(-0.2 pts)**

ii) *(1 point)* The contribution of gravity along the side of the cone has to be equal to the contribution of the centrifugal force. As such, $mg\cos\theta = \frac{mv^2}{R}\sin\theta$, so $v = \sqrt{Rg\cot\theta}$.

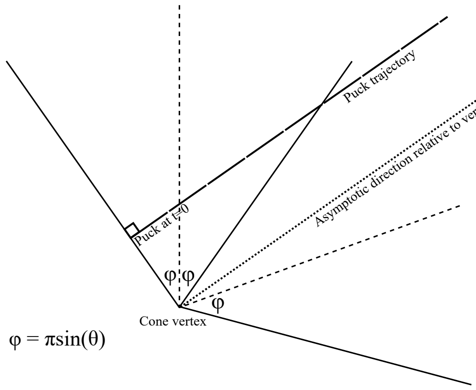
Grading: (preliminary)

- Correct force balance **(0.5 pts)**
- Correct expression for v **(0.5 pts)**
- Mistakes in trigonometry **(-0.1 pts)**

iii) *(2.5 points)* The puck's energy is the sum of its gravitational potential energy and its kinetic energy, so for a maximal vertical displacement h_f, and the speed v_f at that moment, we have $E = \frac{v^2}{8} + gh_f = \frac{v_f^2}{2}$. From the conservation of angular momentum between the initial moment and the minimal distance to axis moment, when the velocity is horizontal, we can deduce that $\frac{v}{2}R = v_f R_f$, where R_f is the minimal distance to the cone's axis. We can relate h_f and R_f through $\frac{R-R_f}{h_f} = \tan\theta$. Then, $h_f = (R - R_f)\cot\theta$. Replacing h_f and v_f in the energy conservation, $\frac{v^2}{8} = \frac{v^2 R^2}{8R_f^2} - g(R - R_f)\cot\theta$. Replacing $v^2 = Rg\cot\theta$ and solving the third degree polynomial in R_f, $8R_f^3 - 9RR_f^2 + R^3 = 0$, yields the roots R, $\frac{1+\sqrt{33}}{16}R$, and some negative value. Our minimal radius is, as such, $\frac{1+\sqrt{33}}{16}R \approx 0.42R$.

Grading: (preliminary)

- Idea of using energy and angular momentum conservation **(0.2 pts)** (0.1 pts for each)
- Correct energy conservation **(0.5 pts)**
- Correct angular momentum conservation **(0.5 pts)**
- Correct relation between h_f and R_f **(0.1 pts)**
- Correct third degree polynomial **(0.5 pts)**
- Picking the physically relevant solution **(0.5 pts)**
- Correct final answer **(0.2 pts)**



iv) *(2.5 points)* Let us unfold the cone and repeat its pattern, as in the figure above. We note that the relation between the apex angle and the angle of the unfolded cone are related through the area of a cone $\pi L^2 \sin\theta = \varphi L^2$ and as such $\varphi = \pi\sin\theta$. The puck's trajectory is a straight line starting perpendicular to the edge of our unfolded cone. The radius vector will rotate, in our drawing, by $90^\circ = \pi/2$ over an infinite period of time. To match the rotation on our representation of the cone with the rotation of the radius vector, we remind ourselves that a $360 = 2\pi$ degree rotation of the radius vector matches a 2φ rotation on our drawing. As such, the number of radians of rotation will be $\frac{\pi}{2}\frac{2\pi}{2\pi\sin\theta} = \frac{\pi}{2\sin\theta}$.

Grading: (preliminary)

- Idea of folding out the cone **(0.2 pts)**
- Stating that the trajectory is a straight line in the folded plane **(1 pts)**
- Correct 90 degree rotation in the folded picture or a correct figure **(0.6 pts)**
- Correct relation between rotation in the folded plane and the rotation of the radial vector **(0.7 pts)**
  - Using θ instead of 2θ as the apex angle **(-0.3 pts)**

Solution 2 by Eppu Leinonen

Let ϕ be the total rotation angle of the radius vector and r its length. The speed related to $\dot\phi$ is $r\dot\phi$ and the speed related to $\dot r$ is $\dot r\csc\phi$, the projection of which to the radial axis is $\dot r$. Thus energy conservation states that $r^2\dot\phi^2 + \dot r\csc\theta = v^2$. Also the projection of the angular momentum onto the symmetry axis is also conserved and as such $r^2\dot\phi = Rv$. From this we solve $\dot\phi$ and substitute to the energy conservation equation to get $\dot r = v\sqrt{1-(R/r)^2}\sin\theta$. Integrating both sides from time $t = 0$ to $t = t$ ($r(0) = R$ and $r(t) = r$) yields $r^2 = \sqrt{R^2 + v^2t^2\sin^2\theta}$. Then $\dot\phi = \frac{Rv}{R^2+v^2t^2\sin^2\theta}$ which we can integrate from $t = 0$ to $t = \infty$ by noticing that $\arctan' x = 1/(1+x^2)$ which gives $\phi_\infty = \frac{\pi}{2\sin\theta}$.

Grading: (preliminary)

- Idea of using energy and angular momentum (projection) **(0.1 pts)** (both required for points)
- Correct energy conservation with respect

to ϕ and r velocity components (can be named freely e.g. v_r and v_ϕ) **(0.4 pts)**

- Correct angular momentum (projection) conservation **(0.4 pts)**
- Equivalent form $\dot{r} = v\sqrt{1-(R/r)^2}$ **(0.5 pts)**
- Solving $r(t)$ (partials below) **(0.5 pts)**
  - Correct integration boundaries **(0.1 pts)**
- Solving ϕ_∞ (partials below) **(0.6 pts)**
  - Setting up integral of the form $\phi_\infty = \int Rv/r^2(t)\mathrm{d}t$ **(0.1 pts)**
  - Correct integration boundaries **(0.1 pts)**
  - $1/\sin\theta$ dependence **(0.3 pts)**

Solution 3 by Eppu Leinonen

Split the velocity into two components: the velocity from changing angle of the radius vector ϕ, $v_\perp$, and the velocity up along the cone $v_\parallel$. In cylindrical coordinates the radial force law now states $N\cos\theta = mv_\perp\dot{\phi} - m\dot{v}_\parallel\sin\theta$ and the vertical force law now states $N\sin\theta = mv_\parallel\cos\theta$. From this we can solve for $\dot{\phi} = \dot{v}_\parallel/(v_\perp\sin\theta)$. From conservation of energy $v_\perp = \sqrt{v^2 - v_\parallel}$ and thus $\phi_\infty = \frac{1}{\sin\theta}\int_0^v \frac{\mathrm{d}v_\parallel}{\sqrt{v^2-v_\parallel^2}} = \frac{\pi}{2\sin\theta}$. The integral can be calculated easily by recoginising that it represents a quarter of the area of a unit circle.

Grading: (preliminary)

- Correct radial and vertical equations of motion **(0.4 pts)** (0.2 for each)
- Representing them in $\phi, v_\parallel, v_\perp$ **(0.4 pts)** (0.2 for each)
- Correct relation between $\dot{\phi}$ and $v_\perp$, $\dot{v}_\parallel$ **(0.5 pts)**
  - For wrong trigonometric dependence **(-0.2 pts)**
- Using the energy conservation law to relate $\dot{\phi}$ to only $v_\perp$ or $v_\parallel$ and derivatives **(0.7 pts)**
- Solving the integral **(0.5 pts)** (partial points below)
  - Correct integration boundaries **(0.1 pts)**
  - Noticing the area of a circle **(0.2 pts)**

Solution 4 by Aleksi Kononen and Eppu Leinonen

Split the velocity into two components: the velocity from changing angle of the radius vector ϕ, $v_\perp$, and the velocity up along the cone $v_\parallel$. Go to the frame corotating with puck. In this frame, there will be three inertial forces on the puck: centrifugal, Coriolis, and Euler. Both the Coriolis and Euler forces point perpendicularly to $v_\parallel$ and tangentially to the cone. However, as the frame is corotating $\dot{v}'_\perp = 0$ in the frame, and thus the Coriolis and Euler forces must cancel. This means that the radial component of the normal force is $N_r = mv_\perp^2/r$. Thus back in the inertial frame we get $\dot{v}_\parallel = \frac{v_\perp^2}{r}\sin\theta$. On the other hand $\dot{\phi} = v_\perp/r = \frac{1}{\sin\theta}\dot{v}_\parallel/v_\perp$. From energy conservation $v_\perp = \sqrt{v^2 - v_\parallel^2}$ and thus $\phi_\infty = \frac{1}{\sin\theta}\int_0^v \frac{dv_\parallel}{\sqrt{v^2 - v_\parallel^2}} = \frac{\pi}{2\sin\theta}$. The integral can be calculated easily by recoginising that it represents a quarter of the area of a unit circle.

Grading: (preliminary)

- Argument for why the radial component of the normal force is equal $mv_\perp^2/r$ **(0.7 pts)** (partials below for non-inertial frame method)
  - Noting that in the inertial frame there are three inertial forces **(0.1 pts)**
  - Noting that Coriolis and Euler forces are parallel to $v_\perp$ **(0.3 pts)**
  - Noting that Coriolis and Euler forces must be equal as $\dot{v}'_\perp = 0$ **(0.3 pts)**
- Noting that the radial component normal force is equal to $mv_\perp^2/r$ **(0.1 pts)**
- Correct relation between $\dot{\phi}$ and $v_\perp$, $\dot{v}_\parallel$ **(0.5 pts)**
  - For wrong trigonometric dependence **(-0.2 pts)**
- Using the energy conservation law to relate $\dot{\phi}$ to only $v_\perp$ or $v_\parallel$ and derivatives **(0.7 pts)**
- Solving the integral **(0.5 pts)** (partial points below)
  - Correct integration boundaries **(0.1 pts)**
  - Noticing the area of a circle **(0.2 pts)**