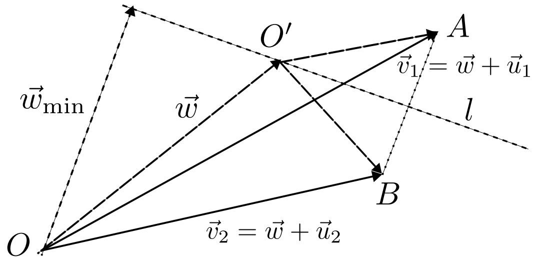
8. Airplanes (7 points) — *Solution by Tudor Plopeanu.*



i) *(1 point)* Let $\vec{w}$ be the speed of the wind at the airplanes' altitude, and let $\vec{u}_1$ and $\vec{u}_2$ be the planes' respective speeds in absence of wind. $\vec{v}_1 = \vec{w} + \vec{u}_1$, $\vec{v}_2 = \vec{w} + \vec{u}_2$. Let O' be the point such that $\vec{OO'} = \vec{w}$, A the point such that $\vec{OA} = \vec{v}_1$, and B the point such that $\vec{OB} = \vec{v}_2$. As $|\vec{w} - \vec{v}_1| = |\vec{w} - \vec{v}_2|$, we know that O' lies on the perpendicular bisector of (AB), which we shall denote as l. A quick check also yields that all points on this line are valid selections for $\vec{w}$. $|\vec{u}_1| = |AO'|$. We can find the minimal airspeed as the length of the perpendicular from A to l, of length $|AB|/2$. For a vector result in terms of $\vec{v}_1$ and $\vec{v}_2$, we have

$$
\begin{aligned}
|\vec{u}_1|_{\min} &= |\vec{v}_1 - \vec{v}_2|/2 \\
&= \frac{1}{2}\sqrt{v_1^2 + v_2^2 - 2v_1 v_2 \cos\alpha}.
\end{aligned}
$$

Grading:

This is relevant for all three subproblems. The solutions are mainly geometric, however several students have tried solving it in a more algebraic manner. If it yielded the correct answer or near correct answer, points were given, but otherwise partial points were not given.

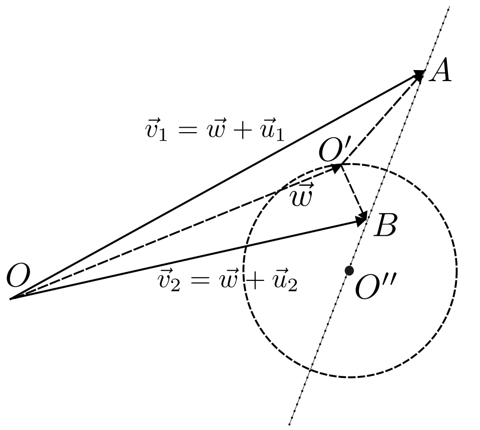
- Representing the problem with vectors and adding the velocities correctly **(0.2 pts)**
- Wrong order or sign **(-0.1 pts)**
- The wind velocity vector w on perpendicular bisector of (AB) **(0.4 pts)**
- The minimum airspeed needs w to lie on the intersection of AB and the perpendicular bisector **(0.1 pts)**
- calculating correct answer **(0.3 pts)**
- Small error but reasonable answer with correct units, or not expanded answer that is simple to expand **(-0.1 pts)**

ii) *(3 points)* The minimal wind speed is given by the length of the perpendicular from O to l. This is the length of $(\vec{v}_1 + \vec{v}_2)/2$ projected on $\vec{v}_1 - \vec{v}_2$, and can be found as

$$\begin{aligned}|\vec{w}|_{\min} &= \frac{|(\vec{v}_1+\vec{v}_2)\cdot(\vec{v}_1-\vec{v}_2)|}{2|\vec{v}_1-\vec{v}_2|}\\ &= \frac{||\vec{v}_1|^2-|\vec{v}_2|^2|}{2|\vec{v}_1-\vec{v}_2|}\\ &= \frac{|v_1^2-v_2^2|}{2\sqrt{v_1^2+v_2^2-2v_1v_2\cos\alpha}}.\end{aligned}$$

Grading:

- Representing the problem with vectors and adding the velocities correctly **(0.5 pts)**
- The wind velocity vector w on perpendicular bisector of (AB) **(0.5 pts)**
- The wind velocity is smallest when it is perpendicular to l **(0.5 pts)**
- Calculating the correct answer **(1.5 pts)**
- Small error but reasonable answer with correct units, or not expanded answer that is simple to expand **(-0.5 pts)**



iii) *(3 points)* In this situation, $|\vec{w}-\vec{v}_1| = 2|\vec{w}-\vec{v}_2|$, so O' is on the Apollonius's circle with respect to the points A and B, such that the points on the circle are 2 times closer to B than to A. Let its center be O''. Its radius is $R_A = 2|\vec{v}_1-\vec{v}_2|/3$ (this can be found by considering the intersection points of the Apollonius circle with the line AB). We are tasked to find the shortest distance from O to this circle, which is found as $|OO''|-R_A$. We note that $\vec{OO''} = (4/3)\vec{v}_2 - (1/3)\vec{v}_1$. The minimal wind speed is thus

$$\begin{aligned}|\vec{w}|_{\min} &= \left|\frac{4}{3}\vec{v}_2 - \frac{1}{3}\vec{v}_1\right| - \frac{2}{3}|\vec{v}_2-\vec{v}_1|\\ &= \frac{\sqrt{17-8\cos\alpha}-4\sin\frac{\alpha}{2}}{3}v.\end{aligned}$$

Grading:

- Representing the problem with vectors and adding the velocities correctly **(0.2 pts)**
- The wind velocity vector w needs to lie on the Apollonius circle (doesn't need the name) **(1.3 pts)**
- if one realises it is a circle, but not the correct one **(0.5 pts)** is given
- Calculating the R_A **(0.5 pts)**
- Calculating the correct answer **(1 pts)**
- Small error but reasonable answer with correct units, or not expanded answer that is simple to expand **(-0.5 pts)**