2. Evaporation (7 points) — *Solution by Jaan Kalda, grading schemes by Mattias Bjerklöv, Marko Tsengov, Eppu Leinonen.*

i) *(2 points)* Water is in a good approximation incompressible; hence, when the piston starts moving, the growing volume must be filled by gas which can be only the water vapours. Thus, the water starts boiling: these vapours must be in equilibrium with water, hence the vapour pressure must be equal to the pressure inside the piston. We can read from the graph that at T_0, the vapour density is $\rho = 420\,\mathrm{g\,m^{-3}}$; this corresponds to the pressure $p_1 = \rho RT/\mu = 70\,\mathrm{kPa}$. With atmospheric pressure $p_0 = 100\,\mathrm{kPa}$, the force

needed to pull the piston is $S(p_0 - p_1) = 300\,\mathrm{N}$.

- Realize that the pressure inside the the cylinder equals the saturated vapour pressure of water at temperature T_0. **0.8 pts**
- Read the density ρ from the graph, in the range $[400, 440]\mathrm{gm^{-3}}$. **0.2 pts**
- Use the ideal gas law to find an expression for the pressure p_1 at temperature T_0. **0.4 pts**
- Correct expression for the force: $S(p_0 - p_1)$. **0.4 pts**
- Correct numerical answer. **0.2 pts**

ii) *(2 points)* When the piston is pulled by displacement a, creating new volume $V_{\text{new}} = S \times a$, the water partially evaporates to fill this volume with vapour and the remaining liquid water cools from temperature T_0 to T_1. The mass of vapour m_v needed to fill the new volume can be calculated using the vapour density $\rho_1 = 405\,\mathrm{g\,m^{-3}}$ as $m_v = Sa\rho_1$. For the heat balance, the energy needed for evaporation must come from the cooling of the remaining liquid water:

$$m_v L = (m - m_v) \cdot c \cdot (T_0 - T_1); \qquad (1)$$

here we have neglected the dependence of L on temperature, and heat capacity of water vapours, because $\mu L \gg 4R(T_1 - T_0)$ (but we have not neglected the work done by piston, because L is actually the enthalpy of evaporation already includes $p\Delta V$). Similarly, since $L \gg c(T_1 - T_0)$, we can neglect m_v in the right-hand-side and express

$$m = \frac{m_v L}{c(T_0 - T_1)} = \frac{\rho_1 S a L}{c(T_0 - T_1)} = 650\,\mathrm{g}. \qquad (2)$$

- Read the vapour density ρ_1 from the graph ($\rho_1 \in [390, 420]\mathrm{gm^{-3}}$). **0.2 pts**
- Correct expression for mass of water vapour. **0.3 pts**
- Correct expression for the latent heat ($m_v L$). **0.3 pts**
- Correct expression for heat lost by water ($(m - m_v) \cdot c \cdot (T_0 - T_1)$.) **0.3 pts**
- Expression for energy conservation. **0.4 pts**
- Correct expression for mass of water m. **0.3 pts**
- Correct numerical answer $m \in [630, 680]\mathrm{g}$ (with correct dimension). **0.2 pts**

iii) *(3 points)* At the thermal equilibrium, there is as much heat flux to the skin as there is heat loss due to evaporation. The former (per area) equals to $\kappa\frac{\mathrm{d}T}{\mathrm{d}x}$ and the latter (per area) — to $-LJm$ where m is the mass of one molecule, which we find to be $m = \mu/N_A$ to get $J\mu L/N_A$. Note that the minus sign comes from the fact that the particles diffuse from higher density areas to lower density areas. Now from the ideal gas law $n = P/Tk_B = PN_A/TR$ to get $J = -D\frac{\mathrm{d}}{\mathrm{d}x}\frac{rp}{Tk_B} = -D\frac{\mathrm{d}}{\mathrm{d}x}\frac{PN_A}{RT}$. Now the pressure of the water vapour is related to r through $P = rp$, where p denotes the saturation pressure of vapour. So,

$$\kappa\frac{\mathrm{d}T}{\mathrm{d}x} = -\frac{DL\mu}{R}\frac{\mathrm{d}}{\mathrm{d}x}\frac{rp}{T},$$

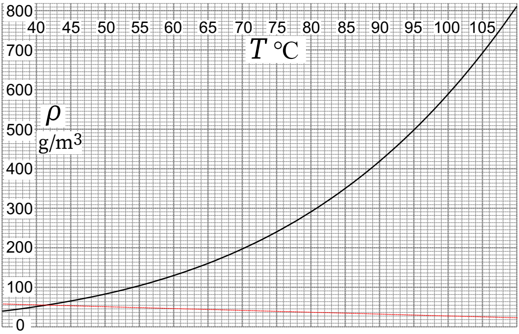
where $p = p(T)$ denotes the water vapour saturation pressure at the local air temperature; hence by integrating over x we obtain

$$\kappa(T - T_s) = \frac{DL\mu}{R}\left[\frac{p(T_s)}{T_s} - \frac{rp(T)}{T}\right],$$

where the index s denotes quantities evaluated at the skin surface. Also, we have used the fact that $r_s = 1$, because at the skin surface, the air is in direct contact with water (due to sweating, skin is wet), so that $pr_s = p(T_s)$. Substituting $\rho = \frac{p\mu}{RT}$ we obtain

$$\rho(T_s) = r\rho(T) + \frac{\kappa}{DL}(T - T_s).$$

Here we evaluate from the graph $r\rho(T) = 24.3\,\mathrm{g\,m^{-3}}$ and $\frac{\kappa}{DL} = 0.51\,\mathrm{g\,m^{-3}\,K^{-1}}$. Now we can draw this straight line onto the graph provided to find the intersection point at $T_s = 41.5\,°\mathrm{C}$.



Grading: (preliminary)

NB! the $\rho = p\mu/RT$ substitution can be done earlier so the schemes below represent only the relevant observations which can be done with ρ already. Also equivalent forms will give points (i.e. if using k and N_a instead of R in the middle steps)

- heat going away from skin (up) = heat going to skin (down) at the equilibrium **0.4 pts**
- Heat flux down $\kappa\frac{\mathrm{d}T}{\mathrm{d}x}$ **0.2 pts**
- Heat flux up magnitude $\frac{DL\mu}{R}\frac{\mathrm{d}}{\mathrm{d}x}\frac{P}{T}$ (partial points available for the equivalents to the steps below) **0.5 pts**
  - Magnitude of heat flux up is LJm **0.3 pts**
  - $m = \mu/N_A$ **0.1 pts**
  - $n = P/Tk_B$ **0.1 pts**
- Deducing that the direction of the heat flow is opposite to $\frac{\mathrm{d}n}{\mathrm{d}x}$ (explicitly mentioned or with the existence of the minus sign in the equations) **0.1 pts**
- $P = rp$ **0.1 pts**
- $\kappa(T - T_s) = \frac{DL\mu}{R}\left[\frac{p(T_s)}{T_s} - \frac{rp(T)}{T}\right]$ (i.e. integrating correctly) **0.4 pts**
  - Or doing a change from $\mathrm{d} \to \Delta$ in the derivatives has to be motivated properly (i.e. for heat conductivity no need for any explicit explanation but for Fick's law one must state that J is constant (due to the amount of particles is conserved)).
- $\rho = p\mu/RT$ **0.1 pts**
- Reading ρ correctly ($\rho_1 \in [800{,}815]\mathrm{gm}^{-3}$)) **0.2p pts**
- Graphical method **0.8 pts**
  - Noticing that $\rho(T_s) = r\rho(T) + \frac{\kappa}{DL}(T - T_s)$ defines a straight line in (T, ρ) **0.8 pts**
  - Any other valid numerical method that is explained is accepted
- Correct final result $T \in [36{,}47]$°C **0.2 pts**

If working with ρ earlier on one can show that that the heat flux up magnitude is

Solution 2 by Eppu Leinonen: One can also work directly with ρ through the fact that $n = N/V = MN_A/\mu V = \rho N_A/\mu$. Then the heat flux magnitude will directly become $LJm = LmD\frac{\mathrm{d}n}{\mathrm{d}x} = LmD\frac{N_A}{\mu}\frac{\mathrm{d}\rho}{\mathrm{d}x} = LD\frac{\mathrm{d}\rho_v}{\mathrm{d}x}$, where ρ_v is the density of the water vapour. Then with correct signs we get

$$\kappa\frac{\mathrm{d}T}{\mathrm{d}x} = -LD\frac{\mathrm{d}\rho_v}{\mathrm{d}x}$$

from which by integrating and using $\rho_v = r\rho$ we get

$$\rho(T_s) = r\rho(T) + \frac{\kappa}{DL}(T - T_s)$$

and the solution proceeds the same way as in solution 1.

The following grading scheme is given to provide exact correspondences to the scheme of solution 1. **Grading:**

- heat going away from skin (up) = heat going to skin (down) at the equilibrium **0.4 pts**
- Heat flux down $\kappa \frac{\mathrm{d}T}{\mathrm{d}x}$ **0.2 pts**
- Heat flux up magnitude $LD\frac{\mathrm{d}\rho_v}{\mathrm{d}x}$ (partial points available for the equivalents to the steps below) **0.6 pts**
  - Working with ρ directly **0.1 pts**
  - Magnitude of heat flux up is LJm **0.3 pts**
  - $n = \rho N_A/\mu$ **0.1 pts**
  - $\mu = mN_A$ **0.1 pts**
- Deducing that the direction of the heat flow is opposite to $\frac{\mathrm{d}n}{\mathrm{d}x}$ (explicitly mentioned or with the existence of the minus sign in the equations) **0.1 pts**
- $\rho_v = r\rho_v$ **0.1 pts**
- $\kappa(T - T_s) = LD(\rho(T_s) - r\rho(T))$ (i.e. integrating correctly) **0.4 pts**
  - Or doing a change from $\mathrm{d} \to \Delta$ in the derivatives has to be motivated properly (i.e. for heat conductivity no need for any explicit explanation but for Fick's law one must state that J is constant (due to the amount of particles is conserved)).
- Reading ρ correctly ($\rho_1 \in [800{,}815]\mathrm{gm}^{-3}$)) **0.2p pts**
- Graphical method **0.8 pts**
  - Noticing that $\rho(T_s) = r\rho(T) + \frac{\kappa}{DL}(T - T_s)$ defines a straight line in (T, ρ) **0.8 pts**
  - Any other valid numerical method that is explained is accepted
- Correct final result $T \in [36{,}47]°\mathrm{C}$ **0.2 pts**