10. Ball magnet (10 points) — *Solution by Jaan Kalda.*

i) *(2 points)* The procedure has two stages: (1) locate the magnetic axis on the ball's surface, and (2) determine which end is N (field exits) and which is S (field enters).

Stage 1: locate the axis. Place the ball-magnet on the flat iron disk. The magnet polarises the iron underneath, creating an attractive image-dipole. The interaction energy is minimised when the magnet's dipole axis is perpendicular to the iron surface (so the two dipoles are aligned head-to-tail). The ball therefore rolls until its axis points straight down at the disk. With a permanent marker, mark a dot at the topmost point of the ball (where the axis exits the upper hemisphere). Then carefully invert the ball, place it on the disk again, and mark a second dot at the new topmost point. The two dots are diametrically opposite and lie on the dipole axis.

For full marking precision, the dot should be placed within $\sim$1 mm of the highest point of the ball; on a 10 mm ball this corresponds to better than $\sim 10^\circ$ angular alignment.

Stage 2: identify which dot is N. At Tallinn's latitude, the geomagnetic field $\vec{B}_E$ points downward into the ground (the Earth's dipole moment points southwards as stated, so its field lines re-enter the ground in the northern hemisphere). A free magnet self-orients with $\vec{\mu}_{\text{magnet}} \parallel \vec{B}_E$. Place the ball on the flat wooden plate (well away from iron objects). The geomagnetic torque rotates the ball until its dipole moment $\vec{\mu}$ aligns with $\vec{B}_E$ — pointing downward, predominantly. The ball does not roll away: $\vec{B}_E$ itself holds the ball in the aligned orientation, since once $\vec{\mu} \parallel \vec{B}_E$ the torque vanishes.

The N pole (where $\vec{B}$ exits, head of $\vec{\mu}$) is then on the bottom of the ball, and the S pole on top. Re-mark the upward-pointing dot with a cross ($\times$): this is the S pole where $\vec{B}$ enters. Leave the downward-pointing dot as a dot: this is the N pole where $\vec{B}$ exits.

Grading.

Method:

- Realising that the magnet rolls so its axis is perpendicular to the iron disk **0.5 pts**
- Recognising that the geomagnetic field at Tallinn points downward into the ground **0.5 pts**
- Realising that the free magnet self-orients with $\vec{\mu} \parallel \vec{B}_E$, hence N points downward **0.5 pts**

Experiment:

- Procedure to mark both poles (dot on top, invert, dot on top again, giving two diametrically opposite axis-points) **0.2 pts**
- Both dots placed within $\sim$1 mm of the true axis-exit points **0.1 pts**
- Correctly marking the upward-pointing dot as a cross (S, field enters) and leaving the downward dot (N, field exits) **0.2 pts**

ii) *(4 points)* **Idea.** Place the ball-magnet between the rails on the wooden plate. As the plate is tilted, gravity tries to roll the ball downhill (rotating $\vec{\mu}$ away from $\vec{B}_E$), while $\vec{B}_E$ exerts a restoring magnetic torque trying to keep $\vec{\mu}$ aligned. At a critical tilt β_c, gravity overcomes the maximum magnetic restoring torque, and the ball begins to roll. Measuring β_c gives B_E.

Setup. The cross on the ball is slightly displaced from the exact top because $\vec{B}_E$ is tilted from vertical (dip angle $\sim 70^\circ$ in Tallinn, so the horizontal component of $\vec{B}_E$ is non-zero and points magnetic-north). Orient the plate so that the cross's small horizontal offset from the topmost point lies parallel to the rails. Equivalently, the

rails should run along the magnetic meridian, with the ball's dipole axis lying in the vertical plane containing $\vec{B}_E$ and the rolling direction.

Procedure.

- Place the plate horizontally and let the ball settle: it self-orients with $\vec{\mu} \parallel \vec{B}_E$, with the cross slightly offset toward magnetic-south of the topmost point.
- Re-orient the plate (in the horizontal plane) so the cross's offset is parallel to the rails (cross equidistant from both rails).
- Slowly tilt one end of the plate upward. Measure the elevation h of the higher end above the lower end with the ruler; for a plate of length L, the tilt angle satisfies $\sin\beta = h/L$.
- Continue tilting smoothly and slowly until the ball just begins to roll down. Record h at that instant.
- Lower the plate, slightly displace the ball to reset, allow re-orientation, repeat. Average over many trials — readings fluctuate due to small variations in initial alignment, friction at the rails, and the smoothness of tilting.

Force/torque balance. Just before rolling, the ball is in static equilibrium between gravity and the magnetic restoring torque. The gravitational torque about the ball's center (acting through the rail contact at distance r) is $\tau_g = mgr\sin\beta$, where $r = d/2$ is the ball radius. The maximum magnetic restoring torque (when $\vec{\mu} \perp \vec{B}_E$) is $\tau_{\mathrm{mag,max}} = \mu B_E$. Critical condition:

$$\mu B_E = mgr\sin\beta_c,$$

$$\boxed{B_E = \frac{mgr\sin\beta_c}{\mu}}.$$

The magnet's dipole moment is $\mu = (4\pi r^3/3)B_\mathrm{r}/\mu_0 = 0.50\,\mathrm{A\cdot m^2}$, with mass $m = \rho_\mathrm{M}\cdot(4\pi r^3/3) \approx 3.93\,\mathrm{g}$. So

$$\frac{mgr}{\mu} = \frac{3.93\times10^{-3}\cdot 9.8\cdot 0.005}{0.50} \approx 385\,\mu\mathrm{T}.$$

Measurements. A representative set of trials with rail length $L = 19\,\mathrm{cm}$:

trial	1	2	3	4	5	6	7	8	9	10	avg
h (mm)	24	26	25	27	23	25	26	24	25	25	25.0

The standard deviation of the readings is $\sigma_h \approx 0.12\,\mathrm{cm}$, giving a standard error on the mean of $\sigma_h/\sqrt{10} \approx 0.04\,\mathrm{cm}$, or about 1.5% of the average.

With $\sin\beta_c = \bar{h}/L = 0.025/0.19 \approx 0.132$:

$$B_E = \frac{mgr\sin\beta_c}{\mu} = 385\,\mu\mathrm{T}\cdot 0.132 \approx 51\,\mu\mathrm{T}.$$

Combined uncertainty (statistical from σ_h plus systematic from μ and r at the few-percent level): $B_E \approx 51(3)\,\mu\mathrm{T}$, in good agreement with the known geomagnetic field at Tallinn's latitude ($\sim$50 μT).

Grading.

- Recognising that gravity drives rolling and magnetic torque resists it **0.2 pts**
- Gravity torque on ball: $\tau_g = mgr\sin\beta$ (force times lever arm r to contact line) **0.2 pts**

- Maximum magnetic torque $\tau_{\mathrm{mag,max}} = \mu B_E$ (when $\vec{\mu} \perp \vec{B}_E$) **0.3 pts**
- Procedure: tilt the plate and record the critical angle when the ball begins to roll **0.4 pts**
- Realising that the rails should be aligned with the cross's horizontal offset (i.e., parallel to the magnetic meridian) **0.4 pts**
- Critical-condition formula $\mu B_E = mgr \sin \beta_c$ **0.5 pts**
- Solving for $B_E = mgr \sin \beta_c / \mu$ **0.3 pts**
- Computing $\mu = (4\pi r^3/3) B_\mathrm{r}/\mu_0$ from the given B_r and ball geometry **0.4 pts**
- Documented measurements (multiple trials, table or list of readings) with averaging and uncertainties **0.8 pts**
- Plausible numerical answer for B_E in the range $30\,\mu\mathrm{T}$ to $70\,\mu\mathrm{T}$ **0.5 pts**

iii) *(4 points)* **Setup.** The titanium piece hangs as a pendulum from a string of length L attached to the stand. Bring the ball-magnet close to the Ti so that one of its poles touches the Ti's surface. The paramagnetic Ti is attracted toward the magnet. Slowly and smoothly pull the magnet-and-Ti combination horizontally; the pendulum tilts. At a critical horizontal displacement x_c, the magnetic attraction can no longer balance the gravitational restoring force, and the Ti detaches from the magnet and swings back. Measuring x_c gives χ.

Practical execution.

- Keep the magnet stuck to the iron pin in the wooden plate as a holder, so it always points one pole forward (rather than rotating in the hand).
- Ensure the magnet's pole touches the Ti at all times and at the same point (the magnet's tip, on the dipole axis); a slight shift up or down reduces the magnetic force.
- Pull very slowly so the system is quasi-static (no inertial effects, $a \approx 0$).
- Repeat many times. Take the *maximum* observed x_c as the result; smaller readings reflect imperfect execution (acceleration, off-axis contact, etc.).

A typical observed range: $x_c \approx 210\,\mathrm{mm}$ to $260\,\mathrm{mm}$.

Physics. The ball-magnet has dipole moment $\mu = (4\pi/3)(d/2)^3 B_\mathrm{r}/\mu_0 \approx 0.50\,\mathrm{A \cdot m^2}$. On its dipole axis, at distance r from the centre,

$$B(r) = \frac{\mu_0 \mu}{2\pi r^3}.$$

At the moment of contact with the Ti, the smallest distance from the magnet's center to the Ti's center is $r_{\min} = (d/2) + d_{\mathrm{Ti}}/2 \approx 5 + 1.2 = 6.2\,\mathrm{mm}$.

The force on the paramagnetic Ti, using the formula given in the problem with $B^2 \propto 1/r^6$ and $|\nabla B^2| = 6B^2/r$:

$$F_{\mathrm{mag}} = \frac{1}{2} \frac{\chi V_{\mathrm{Ti}}}{\mu_0} \cdot 6 \frac{B^2(r)}{r} = \frac{3\chi V_{\mathrm{Ti}} \mu_0 \mu^2}{4\pi^2 r^7}.$$

Force balance at detachment. Just before the Ti detaches, the pendulum's tilt angle θ_c satisfies $\sin\theta_c = x_c/L$, and the horizontal restoring force is $m_{\mathrm{Ti}}\, g \tan\theta_c$. Setting this equal to the maximum magnetic force (at $r = r_{\min}$):

$$m_{\mathrm{Ti}}\, g \tan\theta_c = \frac{3\chi V_{\mathrm{Ti}} \mu_0 \mu^2}{4\pi^2 r_{\min}^7}.$$

Using $m_{\mathrm{Ti}} = \rho_{\mathrm{Ti}} V_{\mathrm{Ti}}$ to cancel V_{Ti}:

$$\chi = \frac{4\pi^2 \rho_{\mathrm{Ti}}\, g \tan\theta_c \cdot r_{\min}^7}{3\mu_0 \mu^2}.$$

Numerical evaluation. Take pendulum length $L = 1.0\,\mathrm{m}$ (estimated from the stand) and $x_c = 235\,\mathrm{mm}$ (representative maximum from many trials). Then $\sin\theta_c = 0.235$, $\tan\theta_c \approx 0.242$. With $\rho_{\mathrm{Ti}} = 4500\ \frac{\mathrm{kg}}{\mathrm{m}^3}$:

$$\chi \approx \frac{4\pi^2 \cdot 4500 \cdot 9.8 \cdot 0.242 \cdot (6.2\times10^{-3})^7}{3 \cdot 4\pi \times 10^{-7} \cdot 0.5^2} \approx \boxed{1.6 \times 10^{-4}}.$$

This is consistent with the literature value for pure titanium ($\chi_{\mathrm{Ti}} \approx 1.8 \times 10^{-4}$), within the precision afforded by the experimental uncertainties.

Grading.

- Setup: pendulum + magnet, identification of detachment as the limit condition **0.3 pts**
- Practical execution detail: keep magnet on iron pin to maintain orientation **0.2 pts**
- Practical execution detail: ensure pole touches Ti at the same axial point **0.2 pts**
- Practical execution detail: pull slowly (quasi-static, $a \approx 0$) **0.2 pts**
- Performing ≥ 5 measurements (either stated explicitly that the reported value is the maximum of ≥ 5 trials, or ≥ 5 measurements documented in a table/list) **0.4 pts**
- Taking the *maximum* (not average) of the measurements as the reported x_c **0.4 pts**
- Combining the given dipole-field and paramagnetic-force formulas into $F_{\mathrm{mag}} = 3\chi V_{\mathrm{Ti}} \mu_0 \mu^2/(4\pi^2 r^7)$ **0.3 pts**
- Identifying $r_{\min}$ as magnet radius + Ti half-thickness (accounting for the wire's radius) **0.4 pts**
- Force balance $mg\tan\theta_c = F_{\mathrm{mag,max}}$ (or equivalently $\sin\theta_c$ if the magnet is pulled perpendicular to the thread rather than strictly horizontally; the two forms differ by $\sim$3% at the critical angle, smaller than experimental scatter; both can be approximated as θ) **0.5 pts**
- Solving this equation for $\chi = 4\pi^2 \rho_{\mathrm{Ti}}\, g \tan\theta_c\, r_{\min}^7/(3\mu_0\mu^2)$ **0.3 pts**
- Documenting the observed maximum θ_c. Full credit for $\theta_c \in [11.9^\circ, 14.5\,\mathrm{mm}]$; reduce by 0.1 pts per half a degree outside this range, down to a minimum of 0 **0.3 pts**
- Numerical answer for χ. Full credit for $\chi \in [1.5 \times 10^{-4}, 1.9 \times 10^{-4}]$; reduce by 0.1 pts for each 0.1×10^{-4} outside this range, down to a minimum of 0 **0.5 pts**

Grading note (default credit for execution details): if the three execution details (iron-pin holder, axial pole contact, slow quasi-static pulling) are not stated explicitly but the reported maximum horizontal displacement is at least 220 mm, award 50% of each item's points (0.1 pts each, keep one decimal). The reasoning: a result in this range can only be obtained if execution was acceptable, even if the student didn't articulate why.