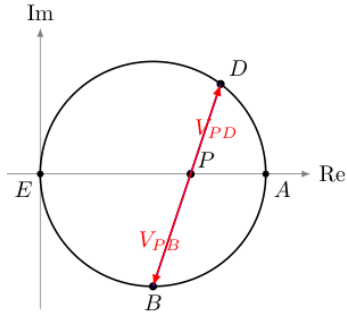
2. Lissajous bridge (6 points) — *Solution by Jaan Kalda.*

Set $V_E = 0$ and $V_A = V$ (phasor). Consider the RC branch: since R_1 and C carry the same current, their voltages are in the impedance ratio $Z_C/Z_{R_1} = 1/(\mathrm{i}\omega R_1 C) = -\mathrm{i}/(\omega R_1 C)$. The factor $-\mathrm{i}$ is a 90° *clockwise* rotation in the phasor plane, so V_{BE} is V_{AB} rotated 90° clockwise — in particular, $V_{AB} \perp V_{BE}$, and V_B lies in the lower half-plane ($\mathrm{Im}\, V_B < 0$).

For the RL branch, $Z_L/Z_{R_2} = \mathrm{i}\omega L/R_2$: the factor i is a 90° *counterclockwise* rotation, so V_{DE} is V_{AD} rotated 90° counterclockwise, giving $V_{AD} \perp V_{DE}$ and V_D in the upper half-plane ($\mathrm{Im}\, V_D > 0$).

By Thales' theorem applied to each branch, both V_B and V_D lie on the circle of diameter $\overline{AE}$ (centre $V/2$, radius $V/2$), on opposite semicircles.



The oscilloscope displays V_{PB} on x and V_{PD} on y; the Lissajous figure degenerates to a line when these phasors are parallel or antiparallel — equivalently, when P, B, D are collinear in the phasor plane. Since V_B and V_D are on opposite sides of the real axis, and V_P is real (a fraction of V), V_P lies *between* them on the line, so V_{PB} and V_{PD} are antiparallel. Hence the displayed line has negative slope and

$$|\tan\alpha| = \frac{|V_{PD}|}{|V_{PB}|}.$$

The potentiometer is split 1:2 from left to right; depending on the orientation this gives $V_P = V/3$ or $V_P = 2V/3$. The power of point P with respect to the Thales circle,

$$\mathrm{pow}(P) = V_P(V_P - V),$$

takes the value $-2V^2/9$ in *both* cases (by symmetry of the quadratic about $V_P = V/2$). Since P is inside the circle and the line BPD cuts it at two points,

$$|PB| \cdot |PD| = |\,\mathrm{pow}(P)| = \frac{2V^2}{9}.$$

Combining with $|\tan\alpha| = |V_{PD}|/|V_{PB}|$:

$$|V_{PB}|^2 |\tan\alpha| = \frac{2V^2}{9},$$

$$\boxed{|V_{PB}| = \frac{V}{3}\sqrt{\frac{2}{|\tan\alpha|}}.}$$

Grading: (6 points total)

- Showing that $V_{AB} \perp V_{BE}$ **0.5 pts**
- Showing that $V_{AD} \perp V_{DE}$ **0.5 pts**

- Correctly identifying that V_B lies in the lower half-plane ($\operatorname{Im} V_B < 0$) ($\operatorname{Im} V_B > 0$ if $V_A = 0$) **0.3 pts**
- Correctly identifying that V_D lies in the upper half-plane ($\operatorname{Im} V_D > 0$) ($\operatorname{Im} V_D < 0$ if $V_A = 0$) **0.3 pts**
- Identifying that V_B lies on a circle with $\overline{AE}$ as a chord **0.3 pts**
- Identifying that V_D lies on a circle with $\overline{AE}$ as a chord **0.3 pts**
- Identifying $\overline{AE}$ as the diameter of the (common) circle **0.3 pts**
- Recognising that Lissajous degeneracy requires B, P, D collinear in the phasor plane **1.5 pts**
- Applying power of a point at V_P to get $|PB| \cdot |PD| = 2V^2/9$ **1.0 pts**
- Correctly relating $|\tan\alpha|$ to the voltage amplitude ratio $|V_{PD}|/|V_{PB}|$ (only if phasors considered) **0.5 pts**
- Combining with the power-of-a-point result to obtain the final answer **0.5 pts**

Since the condition *the line segment makes an angle α with the x-axis* could be understood in different ways, we did not deduct marks for failing to realize that the slope is negative or using $\tan\alpha$ without absolute value.

Alternative solution (pure algebraic/impedance method, for students who do not spot the Thales circle). Ground node E and write $V_A = V$. The voltage dividers at B and D give

$$V_B = \frac{V}{1 + \mathrm{i}\omega R_1 C}, \qquad V_D = \frac{V}{1 - \mathrm{i}R_2/(\omega L)},$$

and the resistive slider gives $V_P = 2V/3$ (or $V/3$; the answer is insensitive to this choice). Introduce dimensionless parameters $u = \omega R_1 C$ and $w = R_2/(\omega L)$, so that

$$V_B = \frac{V(1 - \mathrm{i}u)}{1 + u^2}, \qquad V_D = \frac{V(1 + \mathrm{i}w)}{1 + w^2}.$$

Taking $V_P = 2V/3$, write $V_{PB}/V = X_B + \mathrm{i}Y_B$ and $V_{PD}/V = X_D + \mathrm{i}Y_D$:

$$X_B = \frac{2u^2 - 1}{3(1 + u^2)}, \qquad Y_B = \frac{u}{1 + u^2},$$
$$X_D = \frac{2w^2 - 1}{3(1 + w^2)}, \qquad Y_D = -\frac{w}{1 + w^2}.$$

The Lissajous curve degenerates to a line iff V_{PB}/V_{PD} is real, i.e., $X_B Y_D - Y_B X_D = 0$. Substituting and simplifying yields

$$(u + w)(2uw - 1) = 0,$$

so (since $u, w > 0$) the bridge condition is $uw = 1/2$, i.e., $R_1 R_2 C = L/2$. The slope of the displayed line equals the ratio of the phasors (both now real), conveniently computed from imaginary parts:

$$\tan\alpha = \frac{Y_D}{Y_B} = -\frac{w(1 + u^2)}{u(1 + w^2)}.$$

Substituting $w = 1/(2u)$ gives

$$\tan\alpha = -\frac{2(1+u^2)}{1+4u^2}, \quad \text{so} \quad \frac{1+4u^2}{1+u^2} = \frac{2}{|\tan\alpha|}. \tag{†}$$

The amplitude between B and P follows from $X_B^2 + Y_B^2$:

$$\begin{aligned}\frac{|V_{PB}|^2}{V^2} &= \frac{(2u^2-1)^2 + 9u^2}{9(1+u^2)^2} \\ &= \frac{(4u^2+1)(1+u^2)}{9(1+u^2)^2} = \frac{4u^2+1}{9(1+u^2)}.\end{aligned}$$

Using (†):

$$\boxed{|V_{PB}| = \frac{V}{3}\sqrt{\frac{4u^2+1}{1+u^2}} = \frac{V}{3}\sqrt{\frac{2}{|\tan\alpha|}}\,.}$$

Grading (alternative solution): 6 points total

- Correctly computing V_B as a phasor **0.5 pts**
- Correctly computing V_D as a phasor **0.5 pts**
- Identifying $V_P = 2V/3$ (or $V/3$) **0.3 pts**
- Expressing V_{PB}/V and V_{PD}/V in the form $X + \mathrm{i}Y$ with explicit real and imaginary parts **0.5 pts**
- Recognising that the Lissajous curve degenerates to a line iff V_{PB}/V_{PD} is real (i.e., $X_B Y_D - Y_B X_D = 0$, or equivalent condition) **0.7 pts**
- Correctly deriving the bridge condition $R_1 R_2 C = L/2$ (equivalently $uw = 1/2$) **0.5 pts**
- Expressing $\tan\alpha$ in terms of u and w (or equivalent parameters) **0.5 pts**
- Substituting the bridge condition to write $\tan\alpha$ as a function of u alone **0.5 pts**
- Correctly computing $|V_{PB}|^2$ in terms of u (before factoring) **0.7 pts**
- Recognising the factoring $4u^4 + 5u^2 + 1 = (4u^2+1)(1+u^2)$ that simplifies $|V_{PB}|^2$ **0.8 pts**
- Eliminating u using the $\tan\alpha$ expression **0.3 pts**
- Obtaining the final answer $|V_{PB}| = (V/3)\sqrt{2/|\tan\alpha|}$ **0.2 pts**

Students who reach the correct final form via either route receive full credit. Algebraic errors may lose the corresponding stage points but could retain partial credit if the overall structure is right.