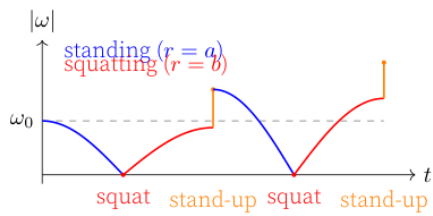
3. Kirill on a swing (8 points) — *Solution by Kaarel Hänni.*

i) *(1.5 points)* Starting at the bottom with $|\omega| = \omega_0$, Kirill (standing, radius a) swings up; $|\omega|$ falls smoothly to 0 at the turning point, after a quarter-period $T_a/4 = (\pi/2)\sqrt{a/g}$. At the turning point Kirill squats (radius $a \to b$, no effect on the plot since $\omega = 0$). Swinging back while squatting, $|\omega|$ rises smoothly from 0 to η_1 (see part c; $\eta_1 = \omega_0\sqrt{a/b} < \omega_0$) after a quarter-period $T_b/4 = (\pi/2)\sqrt{b/g}$, which is *longer* than the first since $b > a$. At the bottom Kirill stands up: $|\omega|$ *jumps* from η_1 to $\omega_1 = \eta_1(b/a)^2 = \omega_0(b/a)^{3/2} > \omega_0$. The second half of the period repeats the pattern with new amplitudes ω_1, $\eta_2 = \omega_1\sqrt{a/b} > \omega_0$, $\omega_2 = \omega_0(b/a)^3$.

The sketch should therefore show:

- smooth arches during swinging, cusps at the bottom (jumps) and smoothness at turning points;
- the "squatting" arches taking longer than the "standing" ones;
- growth of successive peaks.

Part (i): Angular speed vs time, first full period.



Part (i) — qualitative sketch of $|\omega|$ (or signed ω) vs t:

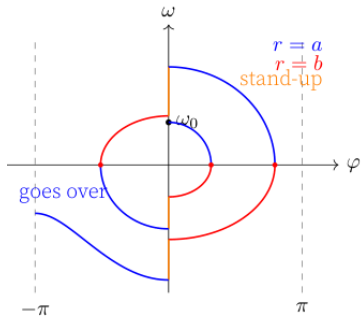
- Approximately sinusoidal arcs between jumps **0.3 pts**
- Instantaneous jumps at the maximum values of ω (at the bottom of the swing, during stand-up) **0.6 pts**
- Smaller amplitude during "squatting" arches, or equivalently a longer quarter-period while $r = b$ **0.3 pts**
- Overall growth of peak ω over the period **0.3 pts**

A sketch of signed ω (rather than $|\omega|$) receives full credit if the curve correctly corresponds to the axis labels and all other qualitative features are preserved. Failure to correctly mark $|\omega|$, where necessary, deducts 0.5 pts. The same goes for not completing the full period of the first swing.

ii) *(2 points)* The trajectory in the (α, ω) plane pieces together arcs of pendulum-like phase curves at two radii, a and b. Each half-swing traces an arc of the energy conservation curve for the relevant radius. Stand-ups at $\alpha = 0$ appear as *vertical segments* where $|\omega|$ jumps from η_i to $\omega_i = \eta_i(b/a)^2$. Squats at the turning points ($\omega = 0$) appear as smooth transitions (the radius-a arc meets the radius-b arc tangentially at $\omega = 0$).

The motion is an outward spiral: each full cycle brings Kirill back to $\alpha = 0$ with a larger $|\omega|$. When ω at the bottom (standing, radius a) first exceeds $\omega_{\min} = 2\sqrt{g/a}$, the next upward arc does not close: it reaches $\alpha = \pi$ with $\omega > 0$ and continues past the top. This is the open curve on the sketch.

Part (ii): Phase diagram (ω vs φ), first three half-swings.



Key features for grading:

- lens-shaped/pendulum-like arcs (not ellipses);
- vertical line segments only at $\alpha = 0$ (stand-ups), smooth meeting at turning points (squats);
- spiral grows outward;
- final open curve goes over the top with non-zero ω.

Failure to use the phase diagram of angular velocity and angle, deducts 0.5 pts from the sum of part ii).

The number of spiral loops is not required to match the actual answer N computed in part (c).

Part (ii) — qualitative sketch of phase diagram ω vs φ:

- Elliptical arcs for small-amplitude (early) cycles, with standing arcs ($r = a$) visibly taller (steeper in ω) than squatting arcs ($r = b$) **0.5 pts**
- Trajectory passes *continuously* through the turning points ($\omega = 0$): no jump at maximum $|\varphi|$ **0.3 pts**
- Vertical jumps at $\varphi = 0$ (during stand-up at the bottom): ω jumps by factor $(b/a)^2$ **0.4 pts**
- Overall growth: each successive cycle encloses the previous one, spiralling outward **0.4 pts**
- Final half-cycle goes over the top — trajectory crosses $\varphi = \pm\pi$ with ω still nonzero (no turning point on this segment) **0.4 pts**

iii) *(4.5 points)* **Growth factor per stand-up.** Let ω_i denote the angular velocity just after the i-th stand-up (at the bottom, standing, radius a); let η_i denote the angular velocity just before that stand-up (at the bottom, squatting, radius b).

Energy conservation on the way up (standing, radius a) to turning point at angle α:

$$\tfrac{1}{2}(\omega_i a)^2 = ga(1-\cos\alpha) \implies \omega_i^2 a = 2g(1-\cos\alpha).$$

At the turning point $\omega = 0$, so the squat ($a \to b$) leaves α and ω unchanged. Swinging back down (squatting, radius b):

$$\tfrac{1}{2}(\eta_{i+1} b)^2 = gb(1-\cos\alpha) \implies \eta_{i+1}^2 b = 2g(1-\cos\alpha).$$

Equating: $\eta_{i+1} = \omega_i\sqrt{a/b}$. The stand-up is a purely radial motion, so angular momentum about the pivot is conserved: $\eta_{i+1} b^2 = \omega_{i+1} a^2$, whence

$$\frac{\omega_{i+1}}{\omega_i} = \sqrt{\frac{a}{b}} \cdot \frac{b^2}{a^2} = \left(\frac{b}{a}\right)^{3/2}.$$

After N stand-ups, $\omega_N = \omega_0 (b/a)^{3N/2}$.

Minimum angular velocity for a loop. The rods are rigid, so the condition for reaching the top is simply that the kinetic energy at the bottom suffice to raise Kirill by $2a$:

$$\tfrac{1}{2}(\omega_{\min} a)^2 = 2ga \implies \omega_{\min} = 2\sqrt{g/a}.$$

Counting. Kirill first loops over when $\omega_N \geq \omega_{\min}$, i.e.

$$N \geq \frac{2}{3} \frac{\ln(\omega_{\min}/\omega_0)}{\ln(b/a)}.$$

Numerical evaluation. With $a = 2.5\,\text{m}$, $b = 3.0\,\text{m}$, $\omega_0 = 1.0\,\text{rad/s}$, $g = 10\,\text{m/s}^2$:

$$b/a = 1.2, \quad (b/a)^{3/2} \approx 1.3145,$$
$$\omega_{\min} = 2\sqrt{10/2.5} = 4\,\text{rad/s},$$
$$N \geq \frac{2}{3} \cdot \frac{\ln 4}{\ln 1.2} = \frac{2}{3} \cdot \frac{1.386}{0.1823} \approx 5.07 \implies \boxed{N = 6}.$$

Check: $\omega_5 = 1.2^{7.5} \approx 3.93\,\text{rad/s}$ (just short); $\omega_6 = 1.2^9 \approx 5.16\,\text{rad/s}$ (exceeds $\omega_{\min}$). So Kirill loops over on the swing-up following his *sixth* stand-up.

Grading (part c):

- Angular momentum conservation during the stand-up **0.7 pts**
- Energy conservation during the swing **0.7 pts**
- Growth factor $(b/a)^{3/2}$ per stand-up **0.7 pts**
- Minimum angular velocity $\omega_{\min}^2 = 4g/a$ for the rigid-rod swing to loop over the top **0.8 pts**
- Correct inequality for the over-the-top condition ($\omega_N \geq \omega_{\min}$ after the Nth stand-up) **0.4 pts**
- Solving the inequality for N using the growth law $\omega_N = \omega_0 (b/a)^{3N/2}$, yielding **0.7 pts**
$$N \geq \log[4g/(a\omega_0^2)]/[3\log(b/a)]$$
- Numerical answer $N = 6$ **0.5 pts**