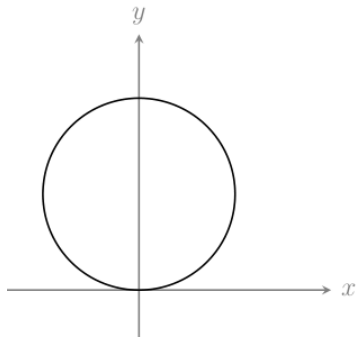
4. Rod and bead (8 points) — *Solutions by Eero Ristolainen, Eppu Leinonen, and Jaan Kalda.*

i) *(1 point) Solution 1 by Eero Ristolainen:* The bead must pass through the origin (rotation axis). Otherwise, the bead would stay on "one side of the rod" and thus at the lowest point of the trajectory the force exerted by the rod couldn't be entirely vertical and thus it couldn't be the lowest point. Hence, the bead must pass through the origin at the lowest point of the trajectory which corresponds to the rod being horizontal. This is enough to make a sketch: the path is a circle of radius R centred at $(0, R)$.



Solution 2: The plot can be made also from solution 4 of part ii.

- Correct sketch **1 pts**
  - Otherwise correct sketch without axis **0.5 pts**.

ii) *(1 point)*

Solution 1 by Eppu Leinonen: The bead's velocity normal to the rod has to be the same as the rod's velocity at that point for the bead to stay on the rod. At the topmost point the bead has no radial velocity. Thus, at the topmost point $v = 2R\omega$. On the other hand, at the topmost point $a = g$ (from Newton's law) and from centripetal acceleration $a = v^2/R$ we can solve $R = g/(4\omega^2)$.

Solution 2 by Eero Ristolainen, Eppu Leinonen: The circular motion must be uniform as the rod performs uniform circular motion. A straight forward formal proof for the this follows from the inscribed angle theorem. The angle of the bead measured from the centre of the rod is twice the angle measured from the origin at the circumference of the circle. The latter changes with speed ω as the corresponding straight line is the rod, thus the former must be 2ω. Thus the centripetal acceleration on the bead is $a = \dfrac{v^2}{R}$. At the topmost point, the speed is $2R\omega$ and the acceleration is only due to gravity and is g. Thus $R = g/(4\omega^2)$.

Grading (solutions 1 and 2 or similar):

- Recognises that the normal velocity of the bead to the rod is the rod's velocity (implicitly or explicitly) OR a valid argument for uniform circular motion with 2ω **0.2 pts**
- $v = 2R\omega$ at the topmost point **0.2 pts**
- At the topmost point $a = g$ **0.2 pts**
- $a = v^2/R$ **0.2 pts**
- Correct answer **0.2 pts**

Solution 3 by Eero Ristolainen: The centre of the motion is $(0, R/2)$ and the angular velocity is 2ω since the bead performs a circle on both sides of the rod. Thus the coordinates of the bead with respect to time are

$$\big(R\sin(2\omega t),\, R\big(\tfrac{1}{2}+\cos(2\omega t)\big)\big) =$$

$$\big(4R\sin(\omega t)\cos(\omega t),\, 4R\sin(\omega t)\sin(\omega t)\big) = \big(f(t)\cos(\omega t),\, f(t)\sin(\omega t)\big),$$

where $f(t) = 4R\sin(\omega t)$ is the location of the bead measured from the axis of rotation (and taking into account the direction). In the frame of the rod, the equation of motion is

$$\ddot{f} = g\sin(\omega t) + \omega^2 f,$$

since Coriolis force is perpendicular to the direction of motion which is counteracted by the rod. Substituting the above expression yields the answer.

Grading:

- A valid argument for uniform circular motion with 2ω **0.2 pts**
- Parametrisation of the circle with respect to angle (θ) and R **0.2 pts**
- Correct equation of motion in the rotating frame **0.2 pts**
- Explains why Coriolis force doesn't matter **0.2 pts**
- Correct answer **0.2 pts**

Solution 4 by Eppu Leinonen, Eero Ristolainen: Go to the corotating frame of the rod. Since the bead is moving in this rotating frame, it experiences a Coriolis force. However, it is directed perpendicular to the velocity which is limited to be along the rod. Hence, it does not affect the equation of motion of the bead and can be ignored. Let r be the coordinate of the bead along the rod (with sign determining the direction). Then the equation of motion along the rod is

$$\ddot{r} = \omega^2 r - g\sin(\omega t).$$

This gives an inhomogeneous second order linear differential equation

$$\ddot{r} - \omega^2 r = -g\sin(\omega t).$$

The general solution is a sum of the general solution to the homogenous problem and any solution to the inhomogeneous problem. The homogenous solution is

$$r_h = Ae^{\omega t} + Be^{-\omega t}.$$

A particular solution we can guess by substituting $C\sin(\omega t)$. This gives an equation for C as $2C\omega^2 = g$ and thus $C = g/(2\omega^2)$. Thus the general solution is

$$r = \frac{g}{2\omega^2}\sin(\omega t) + Ae^{\omega t} + Be^{-\omega t},$$

which can only be periodic (in finite time) if $A = B = 0$, which determines the initial conditions (which of there are only one) for the bead to have a circular trajectory. In the lab frame:

$$\vec{r}(t) = r(t)\,\hat{u}(t) = \frac{g}{2\omega^2}\sin\omega t\,(\cos\omega t, \sin\omega t) = \frac{g}{4\omega^2}(\sin 2\omega t,\, 1-\cos 2\omega t).$$

This is a circle of radius $R = g/(4\omega^2)$ centred at $(0, R)$ (which gives the solution to part i as well).

Grading:

- Correct equation of motion in the rotating frame **0.2 pts**

- States implicitly or explicitly that the general solution to the differential equation is the sum of the homogeneous and particular solutions **0.2 pts**
- Correct homogenous solution **0.1 pts**
- Correct particular solution **0.2 pts**
- States that homogenous solution corresponding to periodic motion is 0 **0.1 pts**
- Correct answer **0.2 pts**

iii) *(2 points) Solution 1 by Jaan Kalda, Eero Ristolainen, Eppu Leinonen* With the spring, the equation of motion becomes

$$\ddot{r} = (\omega^2 - \omega_0^2)\, r - g \sin \omega t, \quad \text{where } \omega_0^2 = k/m.$$

Stability. Perturbations δr from the periodic particular solution obey the homogeneous equation $\delta\ddot{r} = (\omega^2 - \omega_0^2)\,\delta r$. If $\omega_0^2 > \omega^2$ (i.e., $k > m\omega^2$), this is simple harmonic motion at frequency $\sqrt{\omega_0^2 - \omega^2}$ — perturbations remain bounded, orbit is *stable*. If $\omega_0^2 < \omega^2$, perturbations grow exponentially — *unstable*. The stability criterion is

$$k > m\omega^2.$$

The centrifugal term acts as an effective destabilising stiffness $-m\omega^2$, and the spring must overcome it.

As in solution 2 to part ii, we again look at the topmost point of the trajectory. The velocity is again $v = 2R\omega$, but now the acceleration is $a = g + 2Rk$. Solving this gives $R = \dfrac{mg}{2(2m\omega^2 - k)}$. In the case the radius becomes negative, the circular trajectory simply appears below the origin, so we can take the magnitude of the radius above, giving the final result:

$$R = \frac{mg}{2\,|2m\omega^2 - k|}.$$

Grading:

- Equation of motion including the spring **0.2 pts**
- Idea of looking at (small) displacements to analyse stability **0.3 pts**
- Identifying the homogeneous equation governing perturbations **0.5 pts**
- Correct stability criterion $k > m\omega^2$ **0.3 pts**
- Correct equations for the acceleration and velocity at the top of the trajectory **0.5 pts**
- Correct radius formula $R = mg/[2|2m\omega^2 - k|]$ **0.2 pts**

Solution 2 by Jaan Kalda (for finding the radius) The particular solution is again (as in solution 4 to part ii) $r_p = A' \sin \omega t$:

$$\begin{aligned} -\omega^2 A' &= (\omega^2 - \omega_0^2)A' - g \\ \Rightarrow \quad A' &= \frac{g}{2\omega^2 - \omega_0^2} = \frac{mg}{2m\omega^2 - k}. \end{aligned}$$

The corresponding lab-frame orbit is, as in (b), a circle of radius $|A'|/2$, giving the result as in solution 1. For $m\omega^2 < k < 2m\omega^2$ the amplitude A' is positive and the circle sits above the axis (as in parts a, b); at $k = 2m\omega^2$ the driven amplitude diverges (driving resonance — the bounded circular orbit ceases to exist); for $k > 2m\omega^2$ the amplitude is negative, meaning the circle flips to below the axis, but remains stable.
Grading:

- Solving for particular-solution amplitude A' **0.5 pts**
- Correct radius formula $R = mg/[2|2m\omega^2 - k|]$ **0.2 pts**

Solution 2 by Eppu Leinonen Alternatively following solution 4 of part ii, the general solution to this differential equation can be found. However, now the homogenous solution is sum of real exponentials if $\omega^2 - \omega_0^2 > 0$ which is unstable and a sum of sinusoid if $\omega^2 - \omega_0^2 < 0$ which is stable. Then the particular solution can be guessed the same way as in the solution above.

iv) *(4 points) Solution by Eero Ristolainen and Eppu Leinonen*

Now the equation of motion in the frame of the rod is:

$$\ddot{f} = g\sin(\omega t) + a_E \sin((\Omega - \omega)t + \phi) + \omega^2 f.$$

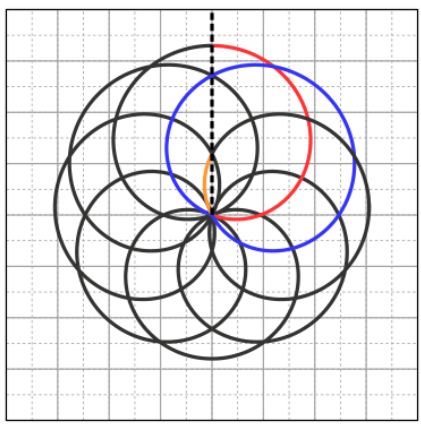
Let us write $\Omega' = \Omega - \omega$ as the angular velocity of the electric field in the frame of the rod. From the figure we see that $f(t)$ is a sum of two periodic components $f(t) = f_1(t) + f_2(t)$. Furthermore, the linearity of the differential equation gives us that

$$\ddot{f}_1 - \omega^2 f_1 = g\sin(\omega t), \qquad \ddot{f}_2 - \omega^2 f_2 = a_E \sin(\Omega' t + \phi).$$

The first one is fulfilled by the exact same solution as before (deduced the same way as in solution 2 of part), $f_1 = \dfrac{g}{2\omega^2}\sin(\omega t)$. We guess the second solution to be of the form $f_2 = A\sin(\Omega' t + \phi)$, which gives $A = a_E/(\Omega'^2 + \omega^2)$.

We see that the solution is periodic. Let this period be T_f. Then we must have that $T_f = nT_{f_1} = pT_{f_2}$ with some positive integers n and p where T_{f_1} and T_{f_2} are the periods of f_1 and f_2 respectively. We notice that the figure is *almost* symmetric with respect to x-axis, which is a huge deviation from the solution with just f_1. Thus the amplitude of f_2 must dominate the one of f_1. Therefore, the sign changes changes of f_2 corresponds to the loops in the figure which there are 8 of. Since the sine function has changes sign twice in a period, we get $4T_{f_2} = T_f = nT_{f_1}$.

For the motion to reach the same ray (from origin, an example dashed in the figure below) twice, it takes time $T_{f_1}/2$ (the rod has made half a turn). From the graph, we can read that in time $\approx 3T_{f_2}/4$ (red + blue) the bead travels less distance than in T_{f_1} (red+blue+orange). Similarly in time $\approx T_{f_2}$ (red + blue + orange + the rest of the black to the horizontal axis) the bead travels more than in T_{f_1} (red + blue + orange).



Thus we get the following inequalities

$$\tfrac{4}{3}T_{f_1} > T_{f_2} > T_{f_1},$$

which gives

$$\tfrac{4}{3} > \tfrac{n}{4} > 1$$

for which $n = 5$ is the only solution. I.e. we get

$$\Omega' = \tfrac{4}{5}\omega \iff \Omega = \tfrac{9}{5}\omega.$$

(This can also be recovered from there being 4 loops corresponding to 5 y-intercepts corresponding to the rod being vertical. This automatically gives $\Omega' = 4/5\,\omega$.)

The symmetry with respect to reflection along the y-axis in the figure (the trajectory is parallel to the x-axis at the extremal points on the y-axis) dictates that the electric field must point vertically when the bead is on the extremal points on the y-axis. Thus, the corresponding displacements along these extremal y-points must align constructively or destructively. I.e. the top-most $|f|_t$ and bottom-most $|f|_b$ distances from the origin correspond to the cases

$$|f|_b = \frac{a_E}{\omega^2 + \Omega'^2} - \frac{g}{2\omega^2}, \qquad |f|_t = \frac{a_E}{\omega^2 + \Omega'^2} + \frac{g}{2\omega^2}.$$

(The formal justification is that at these extremal points $\dot{f}_1 + \dot{f}_2 = 0$ and both contributions are proportional to a sinusoid, thus $\ddot{f}_1$ and $\ddot{f}_2$ are "orthogonal" at the same points. Thus, as gravity is parallel to the rod we have $\ddot{f}_2$ is at its extremum and thus $\dot{f}_1 = 0$. Thus $\dot{f}_2 = 0$ and thus contribution from $\ddot{f}_2$ has to be at its extremum. This implies that the electric field points parallel or anti parallel to the rod at these extremal points. Now $f_1, f_2 \propto \ddot{f}_1, f_2$ (due to both being orthogonal to $\dot{f}_1, \dot{f}_2$) and thus at these extremal points on the y-axis, we have $|f|_b = |f_2| - |f_1|$ and $|f|_t = |f_2| + |f_1|$.)

Measuring

$$|f|_b \approx 2.8d, \qquad |f|_t \approx 3.3d$$

and solving gives

$$g = \frac{d\omega^2}{2}, \qquad a_E = 5d\omega^2.$$

Grading:

- Equation of motion including the electric field **0.1 pts**
- Writing $f(t) = f_1(t) + f_2(t)$ **0.1 pts**
- Deducing $f_1 = \frac{g}{2\omega^2}\sin(\omega t)$ **0.2 pts**
- Guessing $f_2 = A\sin(\Omega' t + \phi)$ **0.2 pts**
- Finding $A = a_E/(\Omega'^2 + \omega^2)$ **0.2 pts**
- Realising f_2 dominates with proper reasoning **0.5 pts**
- Noting periodicity **0.2 pts**
- $T_f = nT_{f_1} = pT_{f_2}$ **0.2 pts**
- Finding a restrictive enough inequality for n (or p) from the figure (or any other procedure that works for finding n) **0.8 pts**
  - For a correct idea (in case it is executed wrong) **0.2 pts**
- $n = 5$ **0.1 pts**
- $\Omega = \frac{9}{5}\omega$ **0.2 pts**
- Deducing with reasonable explanation that electric field points along the y-axis when bead reaches extremal y **0.5 pts**
- Writing $|f|_b$ and $|f|_t$ as the difference/sum of the f_1 and f_2 amplitudes **0.3 pts**
- Solving for g and a_E **0.1 pts**
- $g \approx d\omega^2/2$ and $a_E \approx 5d\omega^2$ **0.3 pts**
  - 0.1pt for one correct.
  - $g \in [0.30, 0.70]\,d\omega^2$, $a_E \in [4.8, 5.2]\,d\omega^2$